# Dynamic Consistency in Ambiguous Dutch Auctions[*]

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**Abstract**

We study a decreasing price auction with an ambiguity-averse seller and two ambiguity-averse buyers. Due to the dynamic structure, buyers learn about the valuation of the opponent buyer during the auction. We characterize a belief formation process that allows buyers to consider their knowledge of the information structure. This process leads to a rectangular ex-ante belief set and implies dynamically consistent behavior. Then, we show that the seller can extract almost all surplus even if buyers behave dynamically consistently. Further, in our setting, buyers accept higher prices compared to a consistent planning approach.

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*JEL subject classification: C73, D81, D83*

## 1    Introduction

In the canonical model with subjective expected utility maximizers and independent private values, it is well-known that the descending price (or Dutch) auction and the first-price sealed-bid auction generate the same equilibrium outcomes. However, this result breaks down if buyers are faced with ambiguous beliefs about their opponents' valuation, see e.g., Bose and Daripa (2009) or Auster and Kellner (2020)).

This paper studies a discrete Dutch auction where buyers have maxmin preferences à la Gilboa and Schmeidler (1989), i.e., buyers maximize their worst-case expected utility. It is well known that ambiguity aversion may lead to dynamically inconsistent behavior. Intuitively, new information might change the worst-case belief such that a player deviates from his ex-ante optimal plan at the interim stage. Thus, usual equilibrium concepts such as sequential equilibrium or perfect Bayesian equilibrium cannot be used to study games with ambiguous beliefs. There exist different approaches to deal with dynamically inconsistent behavior in settings with ambiguous beliefs. Auster and Kellner (2020) and Bose and Daripa (2009) study ambiguous Dutch auction with the consistent planning approach of Siniscalchi (2011). Consistent planning is based on a backward induction

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procedure. Players behave dynamically inconsistently but are sophisticated and consider the preferences of their future selves.

A different approach is rectangularity. Rectangularity is a condition on the ex-ante belief sets. Intuitively, players take into account their future worst-case beliefs and adapt their ex-ante belief sets to ensure dynamically consistent behavior. Sarin and Wakker (1998) and Epstein and Schneider (2003) study rectangular belief sets in decision theoretical setting. Pahlke (2022) extend the definition of rectangularity to multi-stage games with ambiguous incomplete information.

Our setting is similar to the discrete Dutch auction setting of Bose and Daripa (2009). However, we assume that buyers behave dynamically consistent due to rectangular beliefs. The seller wants to sell one indivisible object to one of two buyers. The valuation $v_i \in [0, 1]$ of each buyer is privately known. Further, each buyer faces ambiguity about the valuation of the opposite buyer. The seller starts the auction with a price equal to one and discretely decreasing the price until one of the buyers accepts. More precisely, at the beginning of a period $k$, the seller secretly and randomly chooses one buyer. This buyer is approached first and can accept or reject the offer at a price $p_k$. If he rejects the offer, the other buyer can accept or reject the same offer. If one of the buyers accepts, the game ends immediately. If both buyers reject the price $p_k$, the game proceeds to the next period $k + 1$ with a lower price $p_{k+1}$. The procedure is repeated until one buyer accepts or the last price $p_n$ is reached.

We contribute to the literature in three ways. First, we extend the results of Bose and Daripa (2009) to dynamically consistent behavior. Thus, we show that a seller can extract almost all surplus of almost all buyers even if the buyers behave dynamically consistently. Intuitively, each buyer believes in the worst case that the chance of getting offered the next price is very low. Therefore, the worst-case expected utility of waiting one period is low, and buyers accept higher prices compared to the standard Bayesian setting.

Secondly, since the buyers never know which buyer receives the current offer first, the incomplete information game induced by an MDM cannot be represented by a multi-stage game. Therefore, to study dynamically consistent behavior, we extend rectangularity to an incomplete information game that cannot be modeled as a multi-stage game.

Third, we show that even if the main result of Bose and Daripa (2009) is maintained under dynamically consistent behavior, rectangularity and consistent planning implies different equilibrium strategies. A dynamically consistent buyer takes his future worst-case beliefs into account. The probability of getting the next offered price if he rejects the current price depends on two events. First, the opposite buyer could be asked second and accept the current price offer. Second, in case that the next stage is reached, the opposite buyer could be asked and accept first. A player with rectangular beliefs considers the worst case of both events. Therefore, he behaves even more pessimistically and accepts higher prices than a consistent planner. Thus, this paper is a first step in comparing the implications of rectangularity and consistent planning. Further, to compare our results with Bose and Daripa (2009), we correct a non-trivial error in their original analysis.

This paper is organized as follows: First, we discuss the related literature. In Section 2, we define the modified Dutch mechanism, strategies, and beliefs and formulate the information structure of the game. Then, we define rectangularity and derive the worst-case belief given rectangular beliefs in Section 3. Section 4 proves the surplus extraction

result with rectangular beliefs. The different equilibrium predictions of rectangularity and consistent planning are discussed in Section 5. Finally, Section 6 concludes and discusses future research.

**Related Literature**    There is a vast literature on auction design with expected utility maximizers. Among others Myerson (1981) and Riley and Samuelson (1981) discuss auctions with risk-neutral and Matthews (1983) and Maskin and Riley (1984) with risk-averse buyers.

Karni (1988) shows that first-price sealed-bid auctions and decreasing-price auctions are equivalent if and only if buyers are dynamically consistent. Our result does not contradict the work of Karni (1988). Even if buyers with rectangular beliefs behave dynamically consistently, the rectangular belief sets depend on the information structure. Therefore, in a Dutch auction, rectangularity will lead to different belief sets than in a first-price auction. Dynamically consistent behavior can occur, even if the different auctions are not equivalent.

Lucking-Reiley (1999) provides an interesting field experiment. He compares the revenue of two dynamic auctions (the English and Dutch auctions) with the revenue of two static auctions (first- and second-price auctions). The canonical model with expected utility maximizers predicts that the English auction (Dutch auction) and the second-price auction (first-price auction) are strategically equivalent. In his experiment, the English auction and the second-price auction generate almost the same revenues for the seller. In contrast, the Dutch auction generates 30 percent higher revenues than the first-price auction. This result is in line with our theory.

The literature on ambiguous auctions increased in the last years. Among others, Lo (1998) analyzes first and second-price auctions with ambiguity-averse buyers. Bose et al. (2006) study static auctions with ambiguity-averse buyers and seller. Di Tillio et al. (2016) consider a screening model with one agent and one principle. In their model, ambiguity arises due to an ambiguous mechanism. The principal can design a set of mechanisms and commit to one without revealing it to the buyer. They show that a seller can increase his profit by using an ambiguous mechanism. Lang and Wasser (2022) study a bilateral trade setting with an ambiguity-sensitive buyer and an ambiguity-neutral seller. They show that the buyer can benefit from privately observed ambiguous information about his valuation.

Ghosh and Liu (2021) and Auster and Kellner (2020) investigate sequential auction settings using the consistent planning approach of Siniscalchi (2011). The setting of Ghosh and Liu (2021) differs from our setting. In their model, multiple units of a good are sold to multiple buyers. In each period, each buyer submits a sealed bid simultaneously. The buyer with the highest bid gets one unit of the good and leaves the auction. This procedure is repeated until all units are sold. If two buyers submit the same bid, ties are broken with a coin toss.[1] Similar to our setting Auster and Kellner (2020) analyze a Dutch auction in continuous time. Further, they allow more general belief sets than the $\epsilon$-contamination. In their setting, the seller cannot extract almost all surplus. However, the Dutch auction still generates a higher surplus than a first-price sealed-bid auction. Further, due to the continuous-time structure, the timing becomes less complex, and

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[1]Even if it seems to be a minor difference if ties are broken before or after buyers submit their bits, it influences the equilibrium outcome. We discuss this issue in more detail in Section 6.

consistent planning and rectangular beliefs lead to equivalent equilibrium outcomes in their setting (see Section 6).

# 2 Ambiguous Dutch Auction

Our basic setting and the definition of an ambiguous Dutch auction mostly follow the setting of Bose and Daripa (2009).

## 2.1 Basic Setting

There is one seller who wants to sell one indivisible object. The seller's valuation of this object is normalized to zero. Two buyers with valuation $v_i \in [0,1]$ for $i = 1,2$ compete for the object. The own valuation is private information of each buyer. The seller is risk and ambiguity-neutral. He believes that each buyers valuation is drawn from a distribution $F$ with density $f(v) > 0$ for all $v \in [0,1]$. In contrast, the buyers are less confident about the opponent's valuation. Each buyer is risk-neutral, but ambiguity avers about the valuation of the other buyer and maximizes his worst-case expected payoff à la Gilboa and Schmeidler (1989). The set of priors of each buyer is given by an $\epsilon$-contamination of the density $f$, i.e.,

$$\Phi_i^{-1} := \{g(\cdot) = (1-\epsilon)f(\cdot) + \epsilon l(\cdot) \; : \; l \in \mathcal{P}\},$$

where $\mathcal{P}$ denotes the set of all density functions with support $\text{supp}(l) \subset [0,1]$. Throughout the whole paper, we will denote with capital letter, $F$, $G$, $L$, and $M$ the distribution functions corresponding to densities $f$, $g$, $l$, and $m$. Please note, that the $\epsilon$-contamination structure of $g \in \Phi_i^{-1}$ implies an $\epsilon$-contamination structure of the corresponding distribution function $G(\cdot) = (1-\epsilon)F(\cdot) + \epsilon L(\cdot)$.

## 2.2 The Modified Dutch Mechanism (MDM)

The timing of the modified Dutch mechanism (MDM) is as follows: At the beginning the seller publicly announces a price sequence $\{p_1, p_2, \ldots, p_n\}$, where $p_k$ denotes the asked price in period $k$. In period $k$:

1) The seller secretly tosses a fair coin to decide which buyer to approach first.

2) The buyer chosen in Step 1) gets the offer $p_k$. If he accepts, he receives the object at the price $p_k$, and the game is over.

3) If he rejects, the second buyer gets the offer $p_k$. If he accepts, he receives the object at the price $p_k$, and the game is over.

4) If the second buyer rejects and $k < n$, the game proceeds to period $k + 1$.

This procedure is repeated each period until either one of the buyers accepts a price or period $n$ is reached. If the buyers do not accept any price $p_1, p_2, \ldots, p_n$, the object remains unsold. The procedure and the price sequence are common knowledge, but the

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buyers never know the result of the coin toss at Step 1), i.e., they do not know who is approached first.

We will consider the same price sequence as Bose and Daripa (2009), which depends on the degree of ambiguity $\epsilon$ and a parameter $\delta \in (0,1)$. We will see later that the seller can use $\delta$ to influence the difference between two consecutive prices and, therefore, the surplus of the buyers. For $\delta > 0$, let $\{p_0, p_1, p_2, \ldots, p_n\}$ be the price sequence, where

$$p_0 = 1 \quad \text{and}$$
$$p_k = \frac{(1-\delta)^k}{\left(1 - \delta + \frac{\epsilon\delta}{2}\right)^{k-1}} \quad \text{for any } k > 0.$$

It is important to note that $p_k$ and the price difference

$$\Delta_0 := p_0 - p_1 = \delta,$$
$$\Delta_k := p_k - p_{k+1} = \left(\frac{1-\delta}{1 - \delta + \frac{\epsilon\delta}{2}}\right)^k \frac{\epsilon\delta}{2}$$

are decreasing in $k$. Bose and Daripa (2009) show that $\lim_{n\to\infty} \sum_{k=0}^{n-1} \Delta_k = 1$. Furthermore, for any given $\eta \in (0,1)$ there exists an integer $T$ such that $\sum_{k=0}^{T} \Delta_k \geq 1 - \eta$. Then, let $n$ be the smallest integer for which this inequality is satisfied and $p_n$ the last offered price. This choice of $n$ ensures that all types above $\eta$ participate in the auction. The intuition of the surplus extraction result is as follows: The seller can use $\eta$ and $\delta$ to design the price sequence so that almost all types participate and each type obtains a surplus smaller than $\delta$. Hence, by making $\eta$ and $\delta$ arbitrarily small, the seller can extract almost all surplus.

## 2.3   Buyers' Strategies

Given an MDM, the strategy $\sigma_i : [0,1] \to \{A, R\}^n$ of a buyer $i$ consists of a strategy for each type. A strategy $\sigma^i(v^i)$ of type $v^i$ is a plan to accept or reject the seller's offer at every price offer of the price sequence given the history of the game so far, i.e., $\sigma^i(v^i) \in \{A, R\}^n$, where $A$ denotes accepting and $R$ rejecting.

Ex-post, the seller's payoff is $p_k$ if the object is sold in period $k$ and zero otherwise. The payoff of buyer $i$ of type $v^i$ is $u_i(\sigma_i(v^i), \sigma_j(v^j)) = v^i - p_k$ if he buys the object in period $k$ and zero if he does not obtain the object. We assume that ex-ante, the seller can commit to the mechanism described above, including the price sequence $\{p_1, p_2, \ldots, p_n\}$. Given the mechanism, the setting reduces to an incomplete information game with two players (buyer $i$ and $j$).

Similar to Bose and Daripa (2009), we define interior cut-off strategies as follows.[2]

**Definition 1.** *A strategy of buyer $i$, $i \in \{1, 2\}$, is called an interior cut-off strategy if there exists a vector $\mathbf{v}^i = (v_1^i, \ldots, v_n^i), 0 \leq v_n^i \leq v_{n-1}^i \leq \cdots \leq v_1^i \leq 1$, such that for $k \geq 1$, the highest price accepted by the interval of types $[v_k^i, v_{k-1}^i)$ is $p_k$, where $v_0^i = 1$.*

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[2]Please note, that our definition differs from Bose and Daripa (2009) since we do not require a strict inequality $v_k^i < v_{k-1}^i$ for all $k = 1, \ldots, n$.

We will prove later that without loss of generality, we can restrict to interior cut-off strategies. Furthermore, we assume two simplifying assumptions to solve indifference. If a buyer is indifferent between accepting and rejecting, he accepts the price. Further, if a buyer is indifferent between buying in two different periods, he buys in the earlier period. Thus, buyers choose the seller-preferred action in case of indifference.

## 2.4   Information Structure and Updated Beliefs

As described above, the valuation of each buyer is private information, and buyers are ambiguity-averse. Receiving a price offer $p_k$ reveals information about the valuation of the other buyer. Let us assume that both buyers play interior cut-off strategies. If buyer $i$ gets the price offer $p_k$, he does not know if buyer $j$ already got the offer $p_k$ and rejected it or if he is asked first. Getting the offer $p_k$ only reveals that buyer $j$ rejected all prices before $p_k$. Therefore, the type of buyer $j$ has to be smaller than $v_{k-1}^j$, which is the lowest type who accepts $p_{k-1}$. The information partition of the state space $\Omega = [0, 1]$, which is induced by the MDM and an interior cut-off strategy of the opponent $\mathbf{v}^j$, is as follows

$$
\begin{aligned}
I_{-1}^i &= I_0^i = \Omega, \\
I_1^i &= \left\{ [0, v_0^j), [v_0^j, 1] \right\} = \left\{ [0, 1), \{1\} \right\}, \\
I_k^i &= \left\{ [0, v_{k-1}^j), [v_{k-1}^j, v_{k-2}^j), \dots, [v_1^j, v_0^j), \{1\} \right\},
\end{aligned}
$$

where $k = -1$ denotes the ex-ante stage after the MDM is announced but before the first price is offered.

Assume player $i$ gets the offer $p_k$. To decide whether he should reject or accept the price, he compares the expected payoff of waiting one period and the payoff of accepting $p_k$. The expected payoff of waiting one period depends on the worst-case belief that he gets the offer $p_{k+1}$ given that he rejects $p_k$.

First, we have to specify how ambiguity avers players update their beliefs. We assume that both buyers use prior-by-prior Bayesian updating, i.e., update each belief in $\Phi_i^{-1}$ using Bayes' rule. The prior-by-prior Bayesian update of $\Phi_i^{-1}$ in period $k$ given the event $E \in I_k^i$ is

$$
\Phi_i^{k,E} = \{ \text{Bay}(g|E) \, : \, g \in \Phi_i^{-1} \},
$$

where $\text{Bay}(g|E)$ denotes the Bayesian update of $g$ given the event $E$

$$
\text{Bay}(g|E)(\cdot) = \frac{g(\cdot)}{g(E)}.
$$

Note, that

$$
g_i^{k-1}(E) = 0
$$

for all $E \in \left\{ [v_{l-2}^j, v_{l-3}^j) \right\}_{l=3,\dots,k} \cup \{\{1\}\}$ and $g_i^{k-1} \in \Phi_i^{k-1,E'}$ with $E' \in I_{k-1}^i$. However, these events imply that the game has already ended before period $k$. Therefore, the information sets that influence the decision in period $k$ are only the first two sets of $I_k^i$, i.e., $[0, v_{k-1}^j)$ and $[v_{k-1}^j, v_{k-2}^j)$. Due to the full support assumption on $f$ and since $\epsilon < 1$,

these two information sets have a strictly positive probability for any $g_i^{k-1} \in \Phi_i^{k-1,E'}$ and $E' \in I_{k-1}^i$ and Bayes' rule is well defined.

Formally, a set of interim beliefs exists $\Phi_i^{k,E}$ in period $k$ for each event $E \in I_k^i$. However, for all events except $[0, v_{k-1}^j)$, the game already ended before period $k$. To simplify notation, we sometimes denote with $\Phi_i^k$ the set of interim beliefs given the event $[0, v_{k-1}^j)$.

**Lemma 1.** *The $\epsilon$-contamination structure is maintained under Bayesian updating. Let $E \subseteq \Omega$. Then,*

$$
\begin{aligned}
\Psi^1 &:= \big\{ Bay(g|E) \,:\, g \in \Phi_i^{-1} \big\} \\
&= \big\{ (1 - \epsilon_k) Bay(f|E) + \epsilon_k Bay(l|E) \,:\, l \in \mathcal{P} \big\} =: \Psi^2
\end{aligned} \tag{1}
$$

*with $\epsilon_k = \frac{\epsilon}{(1-\epsilon)f(E)+\epsilon} > \epsilon$ for all $k \geq 1$ and $\epsilon_0 = \epsilon$.*

The proof of Lemma 1 can be found in Section B.0.1 in the Appendix.

**Corollary 1.** *In each period $k \geq 1$, the set of updated beliefs is given by*

$$
\Phi_i^k = \big\{ (1 - \epsilon_k) f^k(\cdot) + \epsilon_k l^k(\cdot) \,:\, l \in \mathcal{P} \big\}
$$

*with*

$$
f^k(\cdot) := \frac{f(\cdot)}{F(v_{k-1}^j)}, \qquad l^k(\cdot) := \frac{l(\cdot)}{L(v_{k-1}^j)}
$$

*and $\epsilon_k = \frac{\epsilon}{(1-\epsilon)F(v_{k-1}^j)+\epsilon}$. Furthermore, $\epsilon_k \geq \epsilon_{k+1}$ for all $k = 1, \ldots n$.*

*Proof.* The result follows immediately from Lemma 1 with $E = [0, v_{k-1}^j)$. Furthermore, $[0, v_k^j) \subset [0, v_{k-1}^j)$ and the full support assumption on $f$ imply $F(v_k^j) < F(v_{k-1}^j)$. Therefore, $\epsilon_k \geq \epsilon_{k+1}$ for all $k = 1, \ldots n$. $\qquad\square$

# 3 Dynamically Consistent Buyers

To decide whether to accept or reject a price offer $p_k$, buyer $i$ compares the expected payoff of waiting one period with his payoff of accepting the current offer. The expected payoff of waiting depends on the worst-case belief that the opponent buyer does not buy the object before buyer $i$ gets the next price offer. The worst-case belief always gives the highest possible probability to the case that the opponent buys the object before buyer $i$ gets the offer $p_{k+1}$. Once buyer $i$ gets the offer $p_{k+1}$, this event can no longer occur, and buyer $i$ updates his beliefs. Then, at period $k$, the worst-case belief of buyer $i$ gives the highest possible probability to the event that the opponent buys the offer before buyer $i$ gets the offer $p_{k+2}$. A dynamically consistent buyer takes this change in the worst-case belief into account.

Epstein and Schneider (2003) define rectangularity (or stability under pasting) as a condition on ambiguous beliefs that ensures dynamically consistent behavior. We will see that rectangular beliefs will lead to different equilibrium strategies as consistent planning. However, we show that the seller can still extract almost all surplus. In Section 5, we

discuss the different implications of rectangularity and the consistent planning approach of Siniscalchi (2011) in more detail.

Now, we define rectangular beliefs for the incomplete information game induced by an MDM. Rectangularity ensures that a buyer takes the possible future worst-case beliefs into account and therefore behaves dynamically consistently. Then, we derive the rectangular worst-case belief of a buyer and discuss surplus extraction with rectangular beliefs.

## 3.1   Rectangularity

The basic idea of rectangularity is that agents take their possible future worst-case beliefs into account. Let us first consider the case without ambiguity, i.e., $\Phi_i^{-1}$ consists of a single prior belief $g$. For an arbitrary event $E \in I_k^i$, $g(E)$ is the marginal probability of the event $E$. Further, remember that $\mathrm{Bay}(g|E)(\cdot)$ denotes the Bayesian update of $g$ given the event $E$. Then, Bayes' rule implies that multiplying the marginal probability $g(E)$ with the updated belief $\mathrm{Bay}(g|E)(\cdot)$ and taking the sum over all events in $I_k^i$, generates the prior belief $g$

$$g(\cdot) = \sum_{E \in I_k^i \,:\, g(E) > 0} g(E) \mathrm{Bay}(g|E)(\cdot).$$

Under ambiguity, the set of beliefs $\Phi_i^{-1}$ is not a singleton. Rectangularity generalizes the above considerations to ambiguous settings. It ensures that the combination of any marginal and updated beliefs is an element of the ex-ante belief set.

**Definition 2.** *The pasting of an ex-ante belief $\bar{g} \in \Phi_i^{-1}$ and a collection of updated beliefs $(g_E)_{E \in I_k^i} \in \bigtimes_{E \in I_k^i} \Phi_i^{k,E}$ is defined as*

$$\bar{g} \circ (g_E)_{E \in I_k^i}(\cdot) := \sum_{E \in I_k^i} \bar{g}(E) g_E(\cdot).$$

*The set of ex-ante beliefs is called **rectangular** (or stable under pasting) if it contains any pasting of an ex-ante belief and interim beliefs, i.e.,*

$$\bar{g} \circ (g_E)_{E \in I_k^i} \in \Phi_i^{-1}$$

*for all $\bar{g} \in \Phi_i^{-1}$ and $(g_E)_{E \in I_k^i} \in \bigtimes_{E \in I_k^i} \Phi_i^{k,E}$.*

An arbitrary ex-ante belief set does not have to satisfy rectangularity. However, Epstein and Schneider (2003) show that given an arbitrary ex-ante belief set $\Phi_i^{-1}$ there always exists a rectangular hull of $\Phi_i^{-1}$, denoted by $\mathrm{rect}(\Phi_i^{-1})$. The rectangular hull is the smallest set of density functions that contains $\Phi_i^{-1}$ and satisfies rectangularity. To construct the rectangular hull, one starts with the set of beliefs at the terminal period $n$ and period $n-1$ and constructs the rectangular hull by combining any marginal probabilities at $n-1$ with any updated belief at $n$. Then, given the rectangular hull in period $n-1$, one proceeds by backward induction. Trivially, if $\Phi_i^{-1}$ is rectangular, then $\Phi_i^{-1} = \mathrm{rect}(\Phi_i^{-1})$.

The following two important properties of rectangularity are also used to construct the rectangular hull. First, the set of Bayesian updates under the rectangular hull $\text{rect}(\Phi_i^{-1})$ equals the set of Bayesian updates under $\Phi_i^{-1}$, i.e., for all $k = 0, \ldots, n$ and $F \in I_k^i$

$$\{\text{Bay}(g|F)(\cdot) \,:\, g \in \Phi_i^{-1}\} = \{\text{Bay}(g|F)(\cdot) \,:\, g \in \text{rect}(\Phi_i^{-1})\}.$$

Second, an analogous statement holds for the set of marginal beliefs $g(F)$ for all $F \in I_k^i$ and $k = -1, \ldots n$ of the rectangular hull $\text{rect}(\Phi_i^{-1})$ and $\Phi_i^{-1}$. Epstein and Schneider (2003) and Pahlke (2022) discuss the construction and properties of the rectangular hull in more detail.

## 3.2 Rectangular Worst-Case Beliefs

The probability $\mathbb{P}^g(A|B)$ of getting the offer $p_{k+1}$ if buyer $i$ rejects the current price $p_k$, depends on two events. First, with positive probability buyer $j$ gets the offer $p_k$ after buyer $i$ rejected $p_k$. If buyer $j$ accepts the offer $p_k$, the game is over, and $i$ does not get the offer $p_{k+1}$. Second, if buyer $j$ rejects $p_k$, period $k + 1$ is reached. In period $k + 1$, with positive probability buyer $j$ gets the offer $p_{k+1}$ first. If he accepts $p_{k+1}$ the game ends and buyer $i$ does not receive the offer $p_{k+1}$. Hence, getting the offer $p_{k+1}$ if buyer $i$ rejects the current price $p_k$ depends on the probability that $j$ does neither accept $p_k$ nor $p_{k+1}$ (given that period $k + 1$ is reached).

Therefore, the worst-case probability $\mathbb{P}^g(A|B)$ depends on the worst-case belief in period $k$ and the worst-case belief in period $k + 1$. However, with ambiguous beliefs, the worst-case belief in period $k + 1$ is generally not the Bayesian update of the worst-case belief in period $k$. Intuitively, in period $k$, if buyer $j$ plays an interior cut-off strategy, the worst-case of buyer $i$ is that buyer $j$ accepts the current price $p_k$. Therefore, the worst-case belief in period $k$ is $g^{*,k}(v^j) = (1-\epsilon_k)f^k(v^j) + \epsilon_k l^k(v^j)$ with $l^k([v_k^j, v_{k-1}^j)) = 1$. Similar, in period $k+1$ the worst-case belief is $g^{*,k+1}(v^j) = (1-\epsilon_{k+1})f^{k+1}(v^j) + \epsilon_{k+1}\bar{l}^{k+1}(v^j)$ with $\bar{l}^{k+1}([v_{k+1}^j, v_k^j)) = 1$. Then, $g^{*,k+1}$ does not equal the Bayesian update of $g^{*,k}$ in period $k + 1$.

The consistent planning approach of Bose and Daripa (2009) and the worst-case belief of ?? assume that the worst-case probability $\mathbb{P}^g(A|B)$ only depends on the worst-case beliefs in period $k$. Therefore, it neither takes into account that $\mathbb{P}^g(A|B)$ depends on the worst-case belief in period $k$ and $k + 1$ nor that the worst-case belief in period $k + 1$ does not equal the Bayesian update of the worst-case belief in period $k$.

Rectangularity takes this change of the worst-case belief into account. The rectangular worst-case probability of getting the offer $p_{k+1}$ if he rejects $p_k$ depends on the worst-case belief of period $k$ and the worst-case belief of period $k + 1$. Therefore, it allows for a change in the worst-case belief between period $k$ and $k + 1$. The following proposition formally characterizes the rectangular worst-case belief if the buyers follow an interior cut-off strategy.

**Proposition 1.** *Suppose the ex-ante belief set of buyer $j$ is given by $rect(\Phi_i^{-1})$. Further, suppose the item has not been sold in periods $1, \ldots, k-1$ and in period $k < n$, the seller offers the item to buyer $i$ at a price $p_k$. Suppose $j$ follows an interior cut-off strategy that gives rise to a vector of cut-offs $\mathbf{v}^j = (v_1^j, \ldots, v_n^j)$. For any type $v$ of $i$, the difference in*

*payoff from buying immediately at a price $p_k$ versus waiting one period to buy at price $p_{k+1}$ is*

$$G_k^i(v) = v - p_k - (v - p_{k+1})\hat{H}_k^i, \tag{2}$$

*where*

$$\hat{H}_k^i = (1 - \epsilon_k)\frac{F^k(v_k^j) + (1 - \epsilon_{k+1})F^k(v_k^j)F^{k+1}(v_{k+1}^j)}{1 + (1 - \epsilon_k)F^k(v_k^j)}. \tag{3}$$

Proposition 1 is important for the equilibrium analysis. To characterize equilibrium strategies, we only have to compare the payoffs from buying immediately with the payoff of waiting one period and accepting the next price. Type $v$ accepts the price $p_k$ if $G_k^i(v) \geq 0$. Since $G_k^i(v)$ is monotone increasing in $v$, a type $v$ who does not accept $p_{k+1}$ would never accept the higher price $p_k$. Thus, we do not need to consider any strategy that involves waiting for more than one period.

*Proof.* The payoff of accepting $p_k$ is $v - p_k$. If buyer $i$ rejects the offer $p_k$, he gets a payoff of $v - p_{k+1}$ if he gets and accepts the offer $p_{k+1}$. If the game ends before he gets the offer $p_{k+1}$, his payoff is zero. Further, $\hat{H}_k^i$ is the probability that $i$ gets the offer $p_{k+1}$ if he rejects $p_k$. Therefore, $G_k^i(v)$ is given by Equation (2). It is left to show that $\hat{H}_k^i$ is specified as above.

In Section A in the Appendix, we show that for a fixed belief $g \in \Phi_i^{-1}$ the probability of getting offer $p_{k+1}$ if he rejects $p_k$ is

$$\mathbb{P}^g(A|B) = \frac{G^k(v_k^j) + G^{k+1}(v_{k+1}^j)G^k(v_k^j)}{1 + G^k(v_k^j)}.$$

The rectangular worst-case belief is then given by

$$\hat{H}_k^i = \min_{g^k \in \text{rect}(\Phi_i^k)} \frac{G^k(v_k^j) + G^{k+1}(v_{k+1}^j)G^k(v_k^j)}{1 + G^k(v_k^j)} \tag{4}$$

where $G^k$ is the distribution function corresponding to $g^k$ and $G^{k+1}$ the distribution function corresponding to the Bayesian update $\text{Bay}(g^k|v < v_k^j)$ at $k + 1$. Since $\Phi_i^{-1} \subseteq \text{rect}(\Phi_i^{-1})$, we first have to show that the distribution functions $G^k$ and $G^{k+1}$ can be represented by an $\epsilon$-contamination of the distribution function $F$.

Note that for any arbitrary $g^k \in \text{rect}(\Phi_i^k)$, $G^k(v_k^j) = g^k(E)$ with $E = [0, v_k^j] \in I_{k+1}^i$ and $G^{k+1}(v_{k+1}^j) = \text{Bay}(g|E)(E')$ with $E' = [0, v_{k+1}^j] \in I_{k+2}^i$. The sets of marginal and updated beliefs given the rectangular hull of $\Phi_i^k$ equal the sets of marginal and updated beliefs given $\Phi_i^k$. Therefore, there exists $m^k \in \Phi_i^k$ and $l^k \in \text{Bay}(\mathcal{P}|E)$ such that

$$G^k(v_k^j) = g^k(E) = m^k(E) = M^k(v_k^j) = (1 - \epsilon_k)F^k(v_k^j) + \epsilon_k L^k(v_k^j), \tag{5}$$

where the last step follows since any density function in $\Phi_i^k$ can be represented as $\epsilon$-contamination of $f$. Similar, for $G^{k+1}$, there exists $\bar{m}^k \in \Phi_i^k$ and $\bar{l}^k \in \text{Bay}(\mathcal{P}|E')$ such that

$$G^{k+1}(v_{k+1}^j) = \text{Bay}(g^k|E)(E') = \text{Bay}(\bar{m}^k|E)(E') = \bar{M}^{k+1}(v_{k+1}^j)$$

$$= (1 - \epsilon_{k+1})F^{k+1}(v_{k+1}^j) + \epsilon_{k+1}\bar{L}^{k+1}(v_{k+1}^j). \tag{6}$$

Now, dividing the denominator and nominator by $G^k(v_k^j)$, we can rewrite Equation (4):

$$\hat{H}_k^i = \min_{g^k \in \text{rect}(\Phi_i^k)} \frac{G^k(v_k^j) + G^{k+1}(v_{k+1}^j)G^k(v_k^j)}{1 + G^k(v_k^j)}$$

$$= \min_{g^k \in \text{rect}(\Phi_i^k)} \frac{1 + G^{k+1}(v_{k+1}^j)}{\frac{1}{G^k(v_k^j)} + 1}$$

$$= \min_{l \in \text{Bay}(\mathcal{P}|E), \bar{l} \in \text{Bay}(\mathcal{P}|E')} \frac{1 + (1 - \epsilon_{k+1})F^{k+1}(v_{k+1}^j) + \epsilon_{k+1}\bar{L}^{k+1}(v_{k+1}^j)}{\frac{1}{(1-\epsilon_k)F^k(v_k^j)+\epsilon_k L^k(v_k^j)} + 1}$$

$$= \frac{1 + (1 - \epsilon_{k+1})F^{k+1}(v_{k+1}^j)}{\frac{1}{(1-\epsilon_k)F^k(v_k^j)} + 1}$$

$$= \frac{(1 - \epsilon_k)F^k(v_k^j) + (1 - \epsilon_k)(1 - \epsilon_{k+1})F^k(v_k^j)F^{k+1}(v_{k+1}^j)}{1 + (1 - \epsilon_k)F^k(v_k^j)}$$

$$= (1 - \epsilon_k)\frac{F^k(v_k^j) + (1 - \epsilon_{k+1})F^k(v_k^j)F^{k+1}(v_{k+1}^j)}{1 + (1 - \epsilon_k)F^k(v_k^j)},$$

where the third step follows from the $\epsilon$-contamination structure of Equation (5) and Equation (6) and the fourth step since $\bar{L}^{k+1}(v_{k+1}^j) = L^k(v_k^j) = 0$ minimizes the equation. $\square$

The following lemma shows that rectangular beliefs lead to dynamically consistent behavior.

**Lemma 2.** *With rectangular beliefs, the worst-case belief in period $k+1$ is the Bayesian update of the worst-case belief in period $k$. Therefore,*

$$\min_{g^k \in rect(\Phi_i^k)} \frac{G^k(v_k^j) + G^{k+1}(v_{k+1}^j)G^k(v_k^j)}{1 + G^k(v_k^j)} = \min_{g^k \in \Phi_i^k, \bar{g}^{k+1} \in \Phi_i^{k+1}} \frac{G^k(v_k^j) + \bar{G}^{k+1}(v_{k+1}^j)G^k(v_k^j)}{1 + G^k(v_k^j)}.$$

*Proof.* First, we show that the left-hand side is greater or equal to the right-hand side. Let

$$g^* \in \arg\min_{g^k \in \text{rect}(\Phi_i^k)} \frac{G^k(v_k^j) + G^{k+1}(v_{k+1}^j)G^k(v_k^j)}{1 + G^k(v_k^j)}.$$

denote a rectangular worst-case belief in period $k$. Further, let $G^{*,k}(v_k^j) = g^*(E)$ with $E = [0, v_k^j] \in I_{k+1}^i$ and $G^{*,k+1}(v_{k+1}^j) = \text{Bay}(g^*|E)(E')$ with $E' = [0, v_{k+1}^j] \in I_{k+2}^i$. Then, similar to the proof of Proposition 1, there exists $m^k, \bar{m}^k \in \Phi_i^k$ such that

$$G^{*,k}(v_k^j) = g^*(E) = m^k(E) = M^k(v_k^j),$$
$$G^{*,k+1}(v_{k+1}^j) = \text{Bay}(g^*|E)(E') = \text{Bay}(\bar{m}^k|E)(E') = \bar{M}^{k+1}(v_{k+1}^j)$$

and

$$\min_{g^k \in \text{rect}(\Phi_i^k)} \frac{G^k(v_k^j) + G^{k+1}(v_{k+1}^j)G^k(v_k^j)}{1 + G^k(v_k^j)} = \frac{M^k(v_k^j) + \bar{M}^{k+1}(v_{k+1}^j)M^k(v_k^j)}{1 + M^k(v_k^j)}$$

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$$\geq \min_{g^k \in \Phi_i^k, \bar{g}^{k+1} \in \Phi_i^{k+1}} \frac{G^k(v_k^j) + \bar{G}^{k+1}(v_{k+1}^j)G^k(v_k^j)}{1 + G^k(v_k^j)}.$$

To prove that the right-hand side is greater or equal to the left-hand side, let

$$(\bar{g}^{*,k}, \bar{g}^{*,k+1}) \in \underset{g^k \in \Phi_i^k, \bar{g}^{k+1} \in \Phi_i^{k+1}}{\arg\min} \frac{G^k(v_k^j) + \bar{G}^{k+1}(v_{k+1}^j)G^k(v_k^j)}{1 + G^k(v_k^j)}.$$

The definition of rectangularity implies that $\bar{g}^{*,k} \circ (\bar{g}_E^{*,k+1})_{E \in I_{k+1}^i} \in \text{rect}(\Phi_i^k)$. Then

$$\min_{g^k \in \Phi_i^k, \bar{g}^{k+1} \in \Phi_i^{k+1}} \frac{G^k(v_k^j) + \bar{G}^{k+1}(v_{k+1}^j)G^k(v_k^j)}{1 + G^k(v_k^j)} \geq \min_{g^k \in \text{rect}(\Phi_i^k)} \frac{G^k(v_k^j) + G^{k+1}(v_{k+1}^j)G^k(v_k^j)}{1 + G^k(v_k^j)}.$$

Combining both directions gives

$$\hat{H}_k^i = \min_{g^k \in \Phi_i^k, \bar{g}^{k+1} \in \Phi_i^{k+1}} \frac{G^k(v_k^j) + \bar{G}^{k+1}(v_{k+1}^j)G^k(v_k^j)}{1 + G^k(v_k^j)}$$
$$= \min_{g^k \in \text{rect}(\Phi_i^k)} \frac{G^k(v_k^j) + G^{k+1}(v_{k+1}^j)G^k(v_k^j)}{1 + G^k(v_k^j)}.$$

$\square$

Hence, the interim worst-case belief is the Bayesian update of the ex-ante worst-case belief due to rectangularity. Therefore, rectangularity implies dynamically consistent behavior. The next subsection shows that the seller can extract almost all surplus, even if the buyers behave dynamically consistently.

# 4    Surplus Extraction under Dynamic Consistency

This section shows that the seller can extract almost all surplus even if the buyers behave dynamically consistently. Thus, the main result of Bose and Daripa (2009) still holds for dynamically consistent buyers. We first characterize and define equilibrium strategies under rectangular beliefs if $\delta$ is sufficiently small. The equilibrium existence follows from the results of Bose and Daripa (2009). In contrast to Bose and Daripa (2009), under rectangular beliefs, there might exist prices such that no type of buyer $i$ or $j$ plans to buy at these prices. However, given the characterization of the equilibrium strategies, we can still show that the seller can extract almost all surplus even if the buyers behave dynamically consistently.

Let us start with the definition and characterization of equilibrium strategies. Remember that a strategy $\sigma_i$ of buyer $i$ specifies for each type a plan to reject or accept the seller's offer at every information set, $\sigma_i : [0, 1] \to \{A, R\}^n$, where $A$ denotes accepting and $R$ rejecting. Further, $u_i(\sigma_i(v^i), \sigma_j(v^j))$ is the ex-post payoff of buyer $i$ of the strategy profile $(\sigma_i, \sigma_j)$ if the types are given by $v^i$ and $v^j$.

**Definition 3.** *A (perfect Bayesian) equilibrium with rectangular beliefs consists of a set of beliefs for each information set and a strategy profile $(\sigma_1^*, \sigma_2^*)$ such that*

- *the ex-ante belief sets equal the rectangular hull $rect(\Phi_i^{-1})$ for $i = 1, 2$,*

- *prior-by-prior Bayesian updating is applied whenever possible, i.e., the set of interim beliefs conditional on the price offer $p_k$ are $rect(\Phi_i^k)$ for $k = 0, \ldots, n$ and $i = 1, 2$, and*

- *given the rectangular belief sets at each information set the equilibrium strategies are a best response for each type, i.e., for all $i \in \{1, 2\}$ and $v^i \in [0, 1]$*

$$\min_{g \in rect(\Phi_i^k)} \mathbb{E}^g(u_i(\sigma_i^*(v^i), \sigma_j^*)) \geq \min_{g \in rect(\Phi_i^k)} \mathbb{E}^g(u_i(\sigma_i(v^i), \sigma_j^*)) \quad \forall \sigma_i(v^i) \in \{A, R\}^n$$

*for all $k = -1, \ldots n$.*

*Further, we call an equilibrium symmetric if $\sigma_i = \sigma_j$.*[3]

So far, our definition of an interior cut-off strategy does not specify any out-off equilibrium behavior. However, as in Bose and Daripa (2009), we can extend an interior cut-off strategy to a perfect cut-off strategy as follows.

**Remark 1.** *Let $p(v)$ denote the highest price that a buyer of type $v$ accepts. By monotonicity, type $v$ would also accept any price smaller than $p(v)$. A perfect cut-off strategy is an interior cut-off strategy with the additional requirement that a type who accepts $p_k$ does accept every price $p_l$ with $l \geq k$.*

The following proposition shows that for $\delta$ sufficiently small, we can restrict the equilibrium analysis without loss of generality to cut-off strategies.

**Proposition 2.** *There exists $\bar{\delta} \in (0, 1)$ such that for all $\delta < \bar{\delta}$ the equilibrium strategies of both players are perfect cut-off strategies and $v_k^i < v_{k-2}^i$ for all $k = 2, \ldots, n$. Furthermore, if no types of $j$ buy at a price $p_k$, then no types of $i$ buy at a price $p_k$ and vice versa.*

The formal proof can be found in Section B.0.2 in the Appendix. To simplify notation, we explain the proof's intuition in interior cut-off strategies. Statements such as "buy $p_k$" mean that the buyer accepts $p_k$ and any lower price but rejects any price higher than $p_k$. Then, one can extend the interior cut-off strategy to a perfect cut-off strategy as described by Remark 1. The idea of the proof is as follows. We first show that there exists a non-degenerate interval of types who buy at a price $p_n$ and a non-degenerate interval of types who buy at $p_1$. Suppose in equilibrium there exists prices $p_{k-l}$ through $p_n$ such that no types of $j$ would buy at prices $\{p_{k-l+1}, \ldots p_{n-1}\}$, but there are types of $j$ who buy at $p_{k-l}$ and, of course, at $p_n$. Then, buyer $i$ will never accept prices in $\{p_{k-l+1}, \ldots p_{n-2}\}$, because he could profit from deviating to price $p_{n-1}$. But, there could exist types of $i$ who buy at $p_{n-1}$. Since no type of buyer $j$ buys at prices $\{p_{k-l+1}, \ldots p_{n-1}\}$, there has to exist a type, $v_{k-l}^j$, who is indifferent between buying at $p_{k-l}$ and waiting till $p_n$.

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[3]Note that $\sigma_i = \sigma_j$ implies that both buyers have the same information partitions $I_k^i = I_k^j$ and therefore the same rectangular belief sets $rect(\Phi_i^{-1}) = rect(\Phi_j^{-1})$ for all $k = -1, \ldots n$.

The proof of Proposition 2 shows that if there are types of $i$ who buy at $p_{n-1}$, then for $\delta$ sufficiently small, there are types of $j$ just below $v_{k-l}^j$ who would profit from deviating and buying at $p_{n-1}$. If no type of $i$ buys at $p_{n-1}$ we can still show that for $\delta$ sufficiently small there are types of $j$ just below $v_{k-l}^j$ who would profit from deviating and buying at $p_{n-1}$ if $l > 2$. Therefore, if no type of $j$ buys at $p_{n-1}$, then there is a non-degenerate interval of types who buy at $p_n$ and a non-degenerate interval of types who buy at $p_{n-2}$. Then, Proposition 2 follows from iteration.

The assumption $l > 2$ is essential. If no type of $i$ buys at $p_{n-1}$ and $l = 2$, the proof of Proposition 2 cannot rule out "price gaps" in the following sense: Let $p_k$ be such a "price gap". Then, there are no types of buyer $i$ or $j$ such that $p_k$ is the highest price accepted by a type of buyer $i$ and $j$. However, "price gaps" can only occur, if $l = 2$. Therefore, a "price gap" contains at most one price, and we get $v_k^i < v_{k-2}^i$. For example, a cut-off strategy such that for both players, there exist types who accept as a price $p_1, p_3, p_5, \ldots,$ and $p_n$ could be an equilibrium. However, a strategy with a "price gap" containing more than one price can never be an equilibrium.

Finally, the existence of "price gaps" does not contradict the definition of perfect cut-off strategies. As described above, the "price gaps" only require that there are no types such that a price $p_k$ is the highest price that these types accept. But it is still possible that there exists types who accept prices $p_{k-l} > p_k$ and therefore also $p_k$.

Given the previous results, we can now characterize equilibrium strategies.

**Proposition 3.** *For $\delta < \bar{\delta}$, in any equilibrium the strategy of any buyer $i$ is a perfect cut-off strategy $\mathbf{v}^i = (v_1^i, \ldots, v_n^i)$, where $v_n^i = p_n$. Further, for $1 \le k \le (n-1)$, $v_k^i \in (p_k, v_{k-1}^i]$, where $v_0 = 1$ and $v_k^i$ is given by*

  *i) if there exist types of $i$ who buy at $p_k$*

$$v_k^i = p_k + \Delta_k \frac{\hat{H}_k^i}{1 - \hat{H}_k^i}$$

  *with $\hat{H}_k^i$ as in Equation (3) of Proposition 1.*

  *ii) $v_k^i = v_{k-1}^i$ if no types of $i$ buy at $p_k$.*

*Proof.* The result follows immediately from Proposition 2, Proposition 1 and Remark 1. Proposition 2 and Remark 1 imply that for sufficiently small $\delta$, any equilibrium strategy is a perfect cut-off strategy as described by Proposition 2 and Remark 1. If no type of buyer $i$ buys at $p_k$, then the interval $[v_k^i, v_{k-1}^i)$ must be degenerated, i.e., $v_k^i = v_{k-1}^i$. If there exists a type who buys at a price $p_k$ and a type who buys at $p_{k-1}$, then $v_k^i$ is the lowest type who accepts $p_k$. This type has to be indifferent between waiting until $p_{k-1}$ and buying at $p_k$, i.e., $G_k^i(v_k^i) = 0$. Furthermore, $G_k^i(p_k) < 0$ and since there exist types who buy at $p_{k-1}$, $G_k^i(v_{k-1}) > 0$. $G_k^i(v)$ is strictly increasing and continuous in $v$. Therefore, there exists $v_k^i \in (p_k, v_{k-1}^i]$ such that $G_k^i(v_k^i) = 0$ and Equation (2) implies

$$v_k^i - p_k = (v_k^i - p_{k+1})\hat{H}_k^i$$

$$\Leftrightarrow \quad v_k^i = p_k + (p_k - p_{k+1})\frac{\hat{H}_k^i}{1 - \hat{H}_k^i} = p_k + \Delta_k \frac{\hat{H}_k^i}{1 - \hat{H}_k^i}.$$

If no type of buyer $i$ buys at $p_{k-1}$, then a type who buys at $p_{k-2}$ has to exist. Thus, the same consideration can be repeated with $v_{k-2}^i = v_{k-1}^i$. $\square$

The existence of a symmetric equilibrium for sufficiently small $\delta$ follows from Bose and Daripa (2009). They show that a symmetric equilibrium exists even if the best response mapping is discontinuous at $v_k^i = v_{k-1}^i$. Hence, a symmetric equilibrium exists for sufficiently small $\delta$ even if "price gaps" with at most one price occur. We focus on symmetric equilibria for the rest of the paper and omit the superscripts $i$ or $j$.

Now, we can state the main result. For any preference parameter $\epsilon$, the seller can design an MDM such that he can extract almost all surplus of almost all buyers.

**Proposition 4.** *For any preference parameter $\epsilon > 0$, there exists $\eta > 0$ and $\delta^*(\epsilon) > 0$ such that for any $\delta < \delta^*(\epsilon)$ there is an MDM such that in any equilibrium of the game induced by this MDM, the item is sold if at least one buyer has a valuation greater than $\eta$ and no type obtains an ex-post surplus greater than $\delta$.*

We have to ensure that the ex-post surplus of both buyers is still smaller than $\delta$ even if "price gaps" occur. The first part of the proof is analogous to Bose and Daripa (2009). The second part considers rectangular beliefs and takes "price gaps" into account.

*Proof.* From the previous results it follows that for any $\epsilon > 0$, there is a $\delta^*(\epsilon) > 0$ such that whenever $\delta < \delta^*(\epsilon)$, an equilibrium exists and all equilibria can be characterized by

$$v_k = p_k + \Delta_k \frac{\hat{H}_k}{1 - \hat{H}_k}.$$

Further, as noted in Section 2.2, for any $\eta \in (0, 1)$, there exists an integer $T$ such that by choosing $n = T$, the price sequence of the MDM covers at least a fraction $(1 - \eta)$ of types. The item is not sold only for types in $[0, \eta]$. Thus, it only remains to show that no type that buys gets an ex-post surplus greater than $\delta$. This part has to be adjusted. For any price $p_k$ with $k > 2$ such that there exist types who accept the price $p_k$, we have one of the following cases.

Case 1) There are types who buy at $p_{k-1}$:[4]

The highest type who accepts $p_k$ is $v_{k-1}$. Therefore, the maximum rent of a buyer who accepts $p_k$ is

$$v_{k-1} - p_k = p_{k-1} - p_k + \Delta_{k-1} \frac{\hat{H}_{k-1}}{1 - \hat{H}k - 1} = \Delta_{k-1} + \Delta_{k-1} \frac{\hat{H}_{k-1}}{1 - \hat{H}_{k-1}}$$

$$< \Delta_{k-1} \left( 1 + \frac{1 - \epsilon}{\epsilon} \right) = \frac{1}{\epsilon} \frac{\delta \epsilon}{2} \left( \frac{1 - \delta}{1 - \delta + \frac{\epsilon \delta}{2}} \right)^{k-1} < \frac{\delta}{2} < \delta,$$

where the first inequality follows since $\hat{H}_{k-1} < 1 - \epsilon$ and the second inequality since $\frac{1 - \delta}{1 - \delta + \frac{\epsilon \delta}{2}} < 1$.

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[4]Case 1) is analogously to Bose and Daripa (2009). However, Bose and Daripa (2009) uses the fact that $\bar{H}_{k-1} < 1 - \epsilon$ which does not hold for the corrected worst-case belief of Section 5.1. Therefore, it is not straightforward that the main result of Bose and Daripa (2009) is still satisfied with the corrected worst-case belief. However, in Section C in the Appendix, we prove that the result still holds by using the specific expression of $\bar{H}_{k-1}$.

Case 2) There are no types who buy at $p_{k-1}$:

Then $v_{k-1} = v_{k-2}$ and similar to Case 1), we get

$$v_{k-2} - p_k = p_{k-2} - p_{k+1} + \Delta_{k-2}\frac{\hat{H}_{k-2}}{1 - \hat{H}_{k-2}} = \Delta_{k-2} + \Delta_{k-1} + \Delta_{k-2}\frac{\hat{H}_{k-2}}{1 - \hat{H}_{k-2}}$$

$$< \Delta_{k-2}\left(2 + \frac{1 - \epsilon}{\epsilon}\right) = \frac{1 + \epsilon}{\epsilon}\frac{\delta\epsilon}{2}\left(\frac{1 - \delta}{1 - \delta + \frac{\epsilon\delta}{2}}\right)^{k-1} < (1 + \epsilon)\frac{\delta}{2} < \delta,$$

where the first inequality follows since, $\Delta_{k-2} > \Delta_{k-1}$ and $\hat{H}_{k-2} < 1 - \epsilon$. The second and third inequality follow since, $\frac{1-\delta}{1-\delta+\frac{\epsilon\delta}{2}} < 1$ and $\epsilon < 1$.

Finally, we have to consider $k = 2$ and $k = 1$. Suppose some types accept $p_2$. Then, by Lemma 5 in Section B.0.2 in the Appendix, it follows that a positive mass of types accepts $p_1$. Similar to Case 1) we get

$$v_1 - p_2 < \delta$$

and the maximum rent of buyers who accept $p_2$ is smaller than $\delta$. For $k = 1$, by definition of $v_0$ and $p_1$, the maximum rent of a type who accepts $p_1$ is

$$v_0 - p_1 = 1 - (1 - \delta) = \delta.$$

Hence, no type can extract a surplus strictly greater than $\delta$. $\qquad\square$

# 5   Rectangularity versus Consistent Planning

Even if our main result generalizes the result of Bose and Daripa (2009) to dynamically consistent players, rectangularity and consistent planning imply different equilibrium strategies. To compare the implications of rectangularity and consistent planning, we first derive the correct worst-case belief in the setting of Bose and Daripa (2009). Then, we illustrate the difference between both approaches with a numerical example, and finally, we explain the difference between both approaches.

## 5.1   Worst-Case Belief of Bose and Daripa (2009)

The entire analysis of Bose and Daripa (2009) is based on the claim that the worst-case belief that player $i$ gets the offer $p_{k+1}$ if he rejects $p_k$ is given by $(1 - \epsilon)H_k^i$, where

$$H_k^i := \frac{F(v_k^j) + F(v_{k+1}^j)}{F(v_k^j) + F(v_{k-1}^j)}.$$

However, this worst-case belief is not correct. Let $A$ denote the event $i$ *obtains the item at $p_{k+1}$* and $B$ the event $i$ *refuses the current offer of $p_k$*. Then, Bose and Daripa (2009) derive $H_k^i$ by claiming that[5]

$$\min_{g^k \in \Phi_i^k} \mathbb{P}^{g^k}(A|B) = \min_{l^k \in \text{Bay}(\mathcal{P}|v < v_{k-1}^j)} (1 - \epsilon)\mathbb{P}^{f^k}(A|B) + \epsilon\mathbb{P}^{l^k}(A|B). \qquad (7)$$

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[5]Please note, that the worst-case expected payoff of waiting equals $\min_{G \in \mathcal{P}_B} \mathbb{P}^g(A|B)(v - p_{k+1}) + (1 - \mathbb{P}^g(A|B)) \cdot 0$. Therefore, minimizing the expected payoff is equivalent to minimizing $\mathbb{P}^g(A|B)$.

But since $A, B \not\subset \Omega$, Lemma 1 and Equation (7) do not hold. The next lemma states the corrected worst-case belief for the consistent planning setting of Bose and Daripa (2009).

**Lemma 3.** *The worst-case belief that player $i$ gets the offer $p_{k+1}$ if he rejects $p_k$ is*

$$\bar{H}_k^i := \frac{(1 - \epsilon)(F(v_k^j) + F(v_{k+1}^j))}{(1 - \epsilon)(F(v_k^j) + F(v_{k-1}^j)) + \epsilon}.$$

*Further, there exists no density $f$ with full support on $[0, 1]$ such that $\bar{H}_k^i = (1 - \epsilon)H_k^i$ for all $k = 1, \ldots, n - 1$.*

Almost all proofs of Bose and Daripa (2009) build upon the fact that the worst-case belief is given by $(1 - \epsilon)H_k^i$ and $H_k^i < 1$ for all $k = 1, \ldots, n - 1$. However, there might exist $k \in 1, \ldots, n - 1$ such that the corrected worst-case belief $\bar{H}_k^i > 1 - \epsilon$. Therefore, correcting the proofs of Bose and Daripa (2009) is not straightforward. A more detailed discussion on the correction of the results of Bose and Daripa (2009) can be found in Section C in the Appendix.

## 5.2  Numerical Example

The following numerical example shows how the different beliefs, the incorrect, the correct, and the rectangular worst-case belief, influence surplus extraction. As in Bose and Daripa (2009), $F$ is the uniform distribution on $[0, 1]$, $n = 7$, $\delta = 0,05$ and $\epsilon = 0.2$. Each table shows the price sequence $p_k$, the equilibrium cut-off values $v_k$, the buyers' maximum rent, and the worst-case belief. The equilibrium cut-off values $v_k$ are derived by solving

$$v_k = p_k + \Delta_k \frac{\tilde{H}_k}{1 - \tilde{H}_k} \tag{8}$$

for $k = 1, \ldots, 6$ with $v_0 = 1$ and $v_7 = p_7$. $\tilde{H}_k$ is either given by the worst-case belief of Bose and Daripa (2009) $(1 - \epsilon)H_k$, the corrected worst-case belief of Section 5.1 $\bar{H}_k$, or the rectangular worst-case belief $\hat{H}_k$.

For all cases, $\tilde{H}_k$ and therefore the cut-off value $v_k$ depend on the cut-off values $v_{k-1}$ and $v_{k+1}$. Then, from Equation (8) we get a system of recursive equations with $v_0 = 1$ and $v_7 = p_7$. Solving this system we get the values $v_1, \ldots, v_6$ listed in Table 1 to Table 3.

Furthermore, the tables show the maximum rent of the buyers. The maximum rent in period $k$ is the highest rent of a type who accepts price $p_k$. More formally, the maximum rent is given by $v_{k-1} - p_k$, i.e., the difference between the highest type who accepts $p_k$ and the price $p_k$.

Table 1 shows the results for the (incorrect) worst-case belief of Bose and Daripa (2009)

$$(1 - \epsilon)H_k = (1 - \epsilon)\frac{F(v_k) + F(v_{k+1})}{F(v_k) + F(v_{k-1})} = (1 - \epsilon)H^{BD}.$$

$H^{BD}$ is smaller than one and therefore $(1 - \epsilon)H_k = (1 - \epsilon)H^{BD}$ smaller than $(1 - \epsilon)$. Bose and Daripa (2009) use this fact repeatedly to prove almost all their results. Table 2

| Price | Value | Max rent | $H^{BD}$ |
|--------|--------|----------|----------|
| 0.9500 | 0.9682 | 0.0500 | 0.9819 |
| 0.9450 | 0.9644 | 0.0232 | 0.9954 |
| 0.9401 | 0.9593 | 0.0243 | 0.9947 |
| 0.9352 | 0.9542 | 0.0241 | 0.9947 |
| 0.9303 | 0.9492 | 0.0240 | 0.9943 |
| 0.9254 | 0.9434 | 0.0238 | 0.9849 |

Table 1: Worst-Case Belief of Bose and Daripa (2009) with $(1 - \epsilon)H_k = (1 - \epsilon)H^{BD}$.

| Price | Value | Max rent | $H^{corr}$ |
|--------|--------|----------|-----------|
| 0.9500 | 0.9866 | 0.0500 | 1.1004 |
| 0.9450 | 0.9823 | 0.0416 | 1.1037 |
| 0.9401 | 0.9768 | 0.0423 | 1.1024 |
| 0.9352 | 0.9715 | 0.0417 | 1.1014 |
| 0.9303 | 0.9655 | 0.0412 | 1.0984 |
| 0.9254 | 0.9562 | 0.0402 | 1.0802 |

Table 2: Correct Worst-Case Belief with $\bar{H}_k = (1 - \epsilon)H^{corr}$.

illustrates the results with the corrected worst-case belief from Section 5.1, i.e.,

$$\bar{H}_k = \frac{(1 - \epsilon)(F(v_k) + F(v_{k+1}))}{(1 - \epsilon)(F(v_k) + F(v_{k-1})) + \epsilon} = (1 - \epsilon)H^{corr}.$$

One can see that $H^{corr}$ is greater than one and thus $\bar{H}_k = (1-\epsilon)H^{corr} > 1-\epsilon$. Therefore, the proofs of Bose and Daripa (2009) cannot be applied to the corrected worst-case belief. However, we show in Section C in the Appendix that the results of Bose and Daripa (2009) still hold with the corrected worst-case belief. Table 2 illustrates this result. The maximum rent of a buyer in Table 2 is still smaller or equal to $\delta$. Table 3 shows the

| Price | Value | Max rent | $H^{rect}$ |
|--------|--------|----------|-----------|
| 0.9500 | 0.9678 | 0.0500 | 0.9771 |
| 0.9450 | 0.9637 | 0.0228 | 0.9883 |
| 0.9401 | 0.9585 | 0.0236 | 0.9867 |
| 0.9352 | 0.9534 | 0.0234 | 0.9856 |
| 0.9303 | 0.9483 | 0.0232 | 0.9842 |
| 0.9254 | 0.9426 | 0.0229 | 0.9752 |

Table 3: Rectangular Worst-Case Belief with $\hat{H}_k = (1 - \epsilon)H^{rect}$.

results with the rectangular worst-case belief

$$\hat{H}_k = (1 - \epsilon_k)\frac{F^k(v_k) + (1 - \epsilon_{k+1})F^k(v_k)F^{k+1}(v_{k+1})}{1 + (1 - \epsilon_k)F^k(v_k)} = (1 - \epsilon)H^{rect}.$$

Comparing Table 2 and Table 3 illustrate the difference of rectangularity and consistent planning. The worst-case belief and the maximum rent are smaller for dynamically consistent buyers than for consistent planning. We compare both approaches in the next section in more detail.

## 5.3 Consistent Planning in the Ambiguous Dutch Auction

Bose and Daripa (2009) analyze the dynamic Dutch auction using the consistent planning approach of Siniscalchi (2011). It is still an open question if settings or conditions exist such that rectangularity and consistent planning lead to the same equilibrium or decision outcomes. Since rectangular beliefs and consistent planning are both based on a backward induction procedure, it is sometimes conjectured for specific settings that both approaches lead to the same equilibrium outcome, e.g., in Auster and Kellner (2020) and Pahlke (2022). However, the numerical example shows that this is not the case in this setting. First, let us describe the consistent planning approach in the ambiguous Dutch auction setting before explaining the difference in equilibrium predictions.

Consistent planning follows a backward induction procedure with an additional tie-breaking rule. Let us start in period $n$ and assume buyer $i$ gets the offer $p_n$. In the last period, the expected payoff of rejecting is zero. Therefore, buyer $i$ accepts the offer if and only if his valuation $v$ satisfies $v \geq p_n$. In period $n-1$, if buyer $i$ gets the offer $p_{n-1}$ he has to compare the following plans: i) accept $p_{n-1}$, ii) reject $p_{n-1}$ and accept $p_n$, iii) reject $p_{n-1}$ and reject $p_n$. Then, consistent planning ensure, that if rejecting $p_n$ is optimal in period $n$, the third plan is optimal in period $n-1$. Further, if accepting $p_n$ is optimal in period $n$, only plan i) and ii) are consistent with the decision in period $n$. Hence, if accepting $p_n$ is optimal in period $n$, buyer $i$ compares plan i) and ii) in period $n-1$ and chooses the one with a higher expected payoff. This procedure is repeated until period zero. In an arbitrary period $k$, buyer $i$ will reject the offer $p_k$ if rejecting $p_{k+1}$ is optimal in period $k+1$. If accepting $p_{k+1}$ is optimal in period $k+1$, in period $k$ buyer $i$ compares the expected payoff of accepting $p_k$ with the expected payoff of rejecting $p_k$ and accepting $p_{k+1}$. Formally, this implies that buyer $i$ accepts $p_k$ if and only

$$G_k^i(v) = v - p_k - (v - p_{k+1})\bar{H}_k^i > 0.$$

However, the procedure described above does not consider any strategic reasoning. To apply the consistent planning approach of Siniscalchi (2011), we have to model the ambiguous Dutch auction as a decision-theoretical setting. For a fixed strategy of buyer $j$, consistent planning analyzes the decision problem of buyer $i$ in periods $n-1$ and $n$ as depicted in Figure 1. Solid circles represent decision nodes of buyer $i$, where $A$ denotes accepting and $R$ rejecting the current price. Empty circles represent nature moves and $E_k^1$ denotes the event that buyer $i$ does not get the offer $p_k$ and $E_k^2$ the event that $i$ gets the offer $p_k$ with $k = n-1, n$. Then, $\bar{H}_k^i$ is given by the worst-case belief of event $E_k^2$.

## 5.4 Consistent Planning and Rectangularity

The decision problem in Figure 1 does not consider that buyer $i$ knows the auction structure. He knows that the worst-case belief of event $E_n^2$ depends on the probability that the object remains unsold at $p_{n-1}$ and the probability that he gets the offer $p_n$ given that period $n$ is reached. Further, buyer $i$ knows that period $n$ is only reached if his worst-case in period $n-1$ does not occur. Taking this into account, Figure 2 represents the true decision setting of buyer $i$, where $q_k^i$ denotes the position of buyer $i$, i.e., if he is asked first or second and $j \in A, B$ the action of buyer $j$. The decision problem, as depicted in Figure 2 allows buyer $i$ to take into account changes of the worst-case belief within the event $E_n^2$. Further, in Figure 2 we can split period $n-1$ and $n$. In

Figure 1: Decision Problem of Buyer $i$ with Consistent Planning in Period $n$ and $n-1$.



Figure 2: Decision Problem of Buyer $i$ with Rectangular Beliefs in Period $n$ and $n-1$.

Figure 1 the switch from period $n-1$ to $n$ happens within the event $E_n^2$. The consistent planning approach of Siniscalchi (2011) is defined for a decision-theoretical setting. It cannot take the specific timing of the ambiguous Dutch auction into account. Therefore, rectangularity and consistent planning lead to different equilibrium strategies. Formally, this follows from the different worst-case beliefs. The rectangular worst-case belief is given by[6]

$$\hat{H}_k^i = \min_{g^k \in \Phi_i^k, g^{k+1} \in \Phi_i^{k+1}} \frac{G^k(v_k^j) + G^{k+1}(v_{k+1}^j)G^k(v_k^j)}{1 + G^k(v_k^j)},$$

whereas the consistent planning worst-case belief is

$$\bar{H}_k^i = \min_{g^k \in \Phi_i^k} \frac{G^k(v_k^j) + G^{k+1}(v_{k+1}^j)G^k(v_k^j)}{1 + G^k(v_k^j)},$$

where for $\bar{H}_k^i$, $G^{k+1}$ is the distribution function corresponding to $g^{k+1} = \text{Bay}(g^k | v < v_k^i)$ the Bayesian update of $g^k$.

For each period $k$, the worst-case belief $g^*$ of buyer $i$ in a period $k$, is that buyer $j$ accepts the current offer, i.e., $g^*(v) = (1-\epsilon)f^k(v) + \epsilon l^k(v)$ with $l^k(v \in [v_{k-1}^i.v_k^i)) = 1$.

---

[6]This representation of the rectangular worst-case belief follows from the proof of Proposition 1.

Therefore, the worst-case belief in period $k+1$ cannot be the Bayesian update of the worst-case belief in period $k$ and for a fixed strategy of buyer $j$

$$\hat{H}_k^i < \bar{H}_k^i.$$

Rectangularity considers this change in the worst-case belief and induces a different optimal strategy than consistent planning.

# 6 Conclusion and Discussion

We consider a Dutch auction mechanism with ambiguity-averse buyers. First, we correct the worst-case belief of Bose and Daripa (2009). Then, we analyze the Dutch auction for dynamically consistent buyers and show that the seller can still extract almost all surplus. Furthermore, we discuss the different implications of rectangularity and consistent planning. Rectangularity allows the buyers to take the structure of the auction into account. Therefore, under the rectangular worst-case belief, a buyer believes that he is less likely to receive the offer $p_{k+1}$ if he rejects $p_k$. This decreases the expected payoff of waiting, and dynamically consistent buyers accept prices closer to their types.

This observation opens many future research questions. As a first step, it is essential to understand how the consistent planning approach of Siniscalchi (2011) could be extended to games. Is Figure 1 the right way to apply consistent planning, or should the definition be extended to take situations as described in Figure 2 into account? Without ambiguity, this problem does not arise since the belief in period $k+1$ is always the Bayesian update of the belief in period $k$.

Further, when do consistent planning and rectangularity predict the same equilibrium outcome? The timing in the Dutch auction setting is very complex. Since the buyers do not know who gets the price offer $p_k$ first, the game cannot be represented by a multi-stage game. Auster and Kellner (2020) analyze an ambiguous Dutch auction in a continuous-time setting. Due to the continuous-time setting, they do not have to specify which buyer is asked first and do not need the complex structure of Bose and Daripa (2009). In their setting, rectangularity and consistent planning induce the same equilibrium strategy.

We conjecture that in multi-stage games, rectangularity and consistent planning will predict equivalent equilibrium outcomes. Multi-stage games require a specific time structure. In each period, players first observe private information (e.g., the move of one opponent from the previous period). Then, players move simultaneously. Therefore, the probability of reaching a specific information set in the next period depends only on the worst-case belief of one period. However, more detailed analyses and discussions are left for future research.

Alternatively, one could use a different time structure for the Dutch auction. In each period, both buyers simultaneously choose an action. If both buyers reject the current price, the game precedes to the next period. If buyer $i$ accepts and buyer $j$ rejects, buyer $i$ gets the object and vice versa. If both buyers accept, the winner is chosen randomly. This version of the Dutch auction mechanism could be modeled as a multi-stage game. However, as discussed by Bose and Daripa (2009), it leads to an uncertain outcome of accepting. Therefore, the full surplus extraction result does not hold in this setting.

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# A    Derivation Worst-Case Belief

The probability that buyer $i$ obtains the item at price $p_{k+1}$ given that he refuses the current offer $p_k$, $\mathbb{P}^g(A|B)$ can be derived analogously to Bose and Daripa (2009) by replacing $F$ with $G$. For completeness, we repeat the derivation here. Let us fix one belief $g \in \Phi_i^{-1}$ and $G$ be the corresponding distribution function. Further, $G^k$ denotes the distribution function corresponding to the updated belief in period $k$, $g^k \in \Phi_i^k$.

We first derive the probability $\pi_k^i$ that period $k+1$ is reached given that $i$ refuses the current offer $p_k$. Then, we derive the probability $\psi_k^i$, that $i$ gets the offer $p_{k+1}$ given that stage $k+1$ is reached. The probability $\mathbb{P}^g(A|B)$ is then given by $\pi_k^i \psi_k^i$.

$\pi_k^i$ can be derived as follows. Let $q^i \in \{1,2\}$ denote the position of player $i$ in period $k$, i.e., if he is asked first or second. Further, let $A^i$ denote the event, that $i$ gets the offer $p_k$. Then,

$$\pi_k^i = \mathbb{P}(q^i = 1|A^i)G^k(v_k^j) + \mathbb{P}(q^i = 2|A^i)$$

with

$$\mathbb{P}(q^i = 1|A^i) = \frac{\mathbb{P}(q^i = 1)\mathbb{P}(A^i|q^i = 1)}{\mathbb{P}(q^i = 1)\mathbb{P}(A^i|q^i = 1) + \mathbb{P}(q^i = 2)\mathbb{P}(A^i|q^i = 2)}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{2}G^k(v_k^j)} = \frac{1}{1 + G^k(v_k^j)}$$

and

$$\mathbb{P}(q^i = 2|A^i) = 1 - \mathbb{P}(q^i = 1|A^i) = \frac{G^k(v_k^j)}{1 + G^k(v_k^j)}.$$

Therefore,

$$\pi_k^i = \frac{2G^k(v_k^j)}{1 + G^k(v_k^j)}.$$

In period $k+1$, $i$ gets the offer $p_{k+1}$ for sure if he is asked first. If he is asked second, he only gets the offer if buyer $j$ refuses $p_{k+1}$:

$$\psi_k^i = \frac{1}{2} + \frac{1}{2}G^{k+1}(v_{k+1}^j).$$

Then,

$$\mathbb{P}^g(A|B) = \pi_k^i \psi_k^i = \frac{2G^k(v_k^j)}{1 + G^k(v_k^j)} \left( \frac{1}{2} + \frac{1}{2}G^{k+1}(v_{k+1}^j) \right)$$

$$= \frac{G^k(v_k^j) + G^k(v_k^j)G^{k+1}(v_{k+1}^j)}{1 + G^k(v_k^j)}.$$

Now, for a fixed $g$, if $G^k(v_k^j) = \frac{G(v_k^j)}{G(v_{k-1}^j)}$ for all $k \geq 1$ we get

$$\mathbb{P}^g(A|B) = \frac{G(v_k^j) + G(v_{k+1}^j)}{G(v_{k-1}^j) + G(v_k^j)}.$$

# B    Proofs

### B.0.1    Proof of Lemma 1

The proof generalizes the proof of Nishimura and Ozaki (2002, page 6 - 8) and consists of four steps. Step 1) proves an observation used recurrently in Steps 2) and 3). Step 2) and 3) show that $\Psi^2 \subseteq \Psi^1$ and $\Psi^1 \subseteq \Psi^2$. Finally, Step 4) proves $\epsilon_k > \epsilon$.

Step 1)    First, we show that $g \in \Psi^1$ if and only if there exists a density function $l \in \mathcal{P}$ such that

$$\begin{aligned}
g(\cdot) &= \frac{(1-\epsilon)f(\cdot) + \epsilon l(\cdot)}{(1-\epsilon)f(E) + \epsilon l(E)} \\
&= \frac{(1-\epsilon)f(E)}{(1-\epsilon)f(E) + \epsilon l(E)} \frac{f(\cdot)}{f(E)} + \frac{\epsilon l(E)}{(1-\epsilon)f(E) + \epsilon l(E)} \frac{l(\cdot)}{l(E)} \\
&= \frac{(1-\epsilon)f(E)}{(1-\epsilon)f(E) + \epsilon l(E)} \mathrm{Bay}(f|E)(\cdot) + \frac{\epsilon l(E)}{(1-\epsilon)f(E) + \epsilon l(E)} \mathrm{Bay}(l|E)(\cdot).
\end{aligned}$$

We can assume, without loss of generality, that $l(E) > 0$. Otherwise, $g(x) = \frac{f(x)}{f(E)}$ for all $x \in E$ which immediately implies that $g \in \Psi^2$.

Step 2)    Let $g$ be an arbitrary density function in $\Psi^2$. Then, the definitions of $\Psi^2$ and $\epsilon_k$ imply that there exists an $l \in \mathcal{P}$ such that

$$\begin{aligned}
g(\cdot) &= ((1-\epsilon_k)\mathrm{Bay}(f|E)(\cdot) + \epsilon_k \mathrm{Bay}(l|E)(\cdot)) \\
&= \frac{(1-\epsilon)f(E)}{(1-\epsilon)f(E) + \epsilon} \mathrm{Bay}(f|E)(\cdot) + \frac{\epsilon}{(1-\epsilon)f(E) + \epsilon} \mathrm{Bay}(l|E)(\cdot). \quad (\text{B.1})
\end{aligned}$$

Since $\mathcal{P}$ consists of all density functions on $[0,1]$ there exists a density function $\bar{l} \in \mathcal{P}$ such that

$$\bar{l}(x) = \mathrm{Bay}(l|E)(x) \quad \forall x \in E$$

and $\bar{l}(x) = 0$ otherwise. Then, $\bar{l}(E) = 1$ and

$$\mathrm{Bay}(\bar{l}|E)(\cdot) = \mathrm{Bay}(l|E)(\cdot).$$

Using this properties of $\bar{l}$ we can rewrite Equation (B.1):

$$g(\cdot) = \frac{(1-\epsilon)f(E)}{(1-\epsilon)f(E) + \epsilon} \mathrm{Bay}(f|E)(\cdot) + \frac{\epsilon\bar{l}(E)}{(1-\epsilon)f(E) + \epsilon\bar{l}(E)} \mathrm{Bay}(\bar{l}|E)(\cdot).$$

Then by Step 1) it follows that $g \in \Psi^1$. Hence, $\Psi^2 \subseteq \Psi^1$.

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Step 3) For the other direction, i.e., $\Psi^1 \subseteq \Psi^2$, let $g$ be an arbitrary element of $\Psi^1$. Then, Step 1) implies

$$
\begin{aligned}
g(\cdot) =& \frac{(1-\epsilon)f(E)}{(1-\epsilon)f(E)+\epsilon l(E)}\text{Bay}(f|E)(\cdot) + \frac{\epsilon l(E)}{(1-\epsilon)f(E)+\epsilon l(E)}\text{Bay}(l|E)(\cdot) \\
=& \frac{(1-\epsilon)f(E)\text{Bay}(f|E)(\cdot)}{(1-\epsilon)f(E)+\epsilon l(E)}\frac{(1-\epsilon)f(E)+\epsilon+\epsilon l(E)-\epsilon l(E)}{(1-\epsilon)f(E)+\epsilon} \\
&+ \frac{\epsilon l(E)\text{Bay}(l|E)(\cdot)}{(1-\epsilon)f(E)+\epsilon l(E)}\frac{(1-\epsilon)f(E)+\epsilon}{(1-\epsilon)f(E)+\epsilon} \\
=&(1-\epsilon)f(E)\text{Bay}(f|E)(\cdot)\frac{(1-\epsilon)f(E)+\epsilon l(E)+\epsilon(1-l(E))}{((1-\epsilon)f(E)+\epsilon l(E))((1-\epsilon)f(E)+\epsilon)} \\
&+ \underbrace{\frac{\epsilon}{(1-\epsilon)f(E)+\epsilon}}_{=\epsilon_k}\frac{((1-\epsilon)f(E)+\epsilon)l(E)\text{Bay}(l|E)(\cdot)}{(1-\epsilon)f(E)+\epsilon l(E)} \\
=&\epsilon_k\underbrace{\frac{((1-\epsilon)f(E)+\epsilon)l(E)}{(1-\epsilon)f(E)+\epsilon l(E)}}_{:=\tilde{\epsilon}}\text{Bay}(l|E)(\cdot) + (1-\epsilon)f(E)\text{Bay}(f|E)(\cdot) \\
&\cdot\left(\frac{1}{(1-\epsilon)f(E)+\epsilon} + \frac{\epsilon(1-l(E))}{((1-\epsilon)f(E)+\epsilon l(E))((1-\epsilon)f(E)+\epsilon)}\right) \\
=&\epsilon_k\tilde{\epsilon}\text{Bay}(l|E)(\cdot) + \underbrace{\frac{(1-\epsilon)f(E)}{(1-\epsilon)f(E)+\epsilon}}_{=1-\epsilon_k}\text{Bay}(f|E)(\cdot) \\
&+ \underbrace{\frac{\epsilon}{(1-\epsilon)f(E)+\epsilon}}_{=\epsilon_k}\frac{(1-l(E))(1-\epsilon)f(E)\text{Bay}(f|E)(\cdot)}{((1-\epsilon)f(E)+\epsilon l(E))} \\
=&\epsilon_k\tilde{\epsilon}\text{Bay}(l|E)(\cdot) + (1-\epsilon_k)\text{Bay}(f|E)(\cdot) \\
&+ \epsilon_k\underbrace{\frac{(1-l(E))(1-\epsilon)f(E)\text{Bay}(f|E)(\cdot)}{((1-\epsilon)f(E)+\epsilon l(E))}}_{=1-\tilde{\epsilon}}\text{Bay}(f|E)(\cdot) \\
=&(1-\epsilon_k)\text{Bay}(f|E)(\cdot) + \epsilon_k\Big(\tilde{\epsilon}\text{Bay}(l|E)(\cdot) + (1-\tilde{\epsilon})\text{Bay}(f|E)(\cdot)\Big).
\end{aligned}
$$

Since $\tilde{\epsilon} \in (0,1]$, $\bar{l}(x) := \tilde{\epsilon}\text{Bay}(l|E)(x) + (1-\tilde{\epsilon})\text{Bay}(f|E)(x)$ for all $x \in E$ and zero otherwise is a density function, i.e., $\bar{l} \in \mathcal{P}$. Furthermore, $\text{Bay}(\bar{l}|E) = \bar{l}$, since $\bar{l}(E) = 1$. Hence, $g(\cdot) \in \Psi^2$ and $\Psi^1 \subseteq \Psi^2$.

Step 4) The full support assumption of $f$ implies that $f(E) < 1$ for all $E \neq \Omega$. Therefore, it follows immediately that

$$
\epsilon_k = \frac{\epsilon}{(1-\epsilon)f(E)+\epsilon} > \epsilon.
$$

Furthermore, if $E = \Omega$, then $f(\Omega) = 1$ and $\epsilon_k = \epsilon$.

### B.0.2 Proof of Proposition 2

The proof follows a similar idea as the proof of Bose and Daripa (2009). However, since most parts explicitly use the form of the worst-case belief, we have to adjust parts of the proof. It consists of several lemmata. First, Lemma 4 shows that both buyers have a positive measure of types who plan to buy at $p_n$. Lemma 5 proves that in any equilibrium, a positive measure of types plans to buy at $p_1$. Then, the idea is as follows. Suppose there is a "price gap" in the sense that there are no types of buyer $j$ who plan to buy at prices in $\{p_{n-l+1}, \ldots, p_{n-1}\}$. Then, there are no types of $i$ who plan to buy at prices $\{p_{n-l+1}, \ldots, p_{n-2}\}$. Lemma 7 shows, that if there exists types of $i$ who buy at $p_{n-1}$, then there exists types of $j$ who strictly prefer to buy at $p_{n-1}$ as well. Further, if no type of $i$ buys at $p_{n-1}$, then there still exist types of $j$ who strictly prefer to buy at $p_{n-1}$ if $l > 2$. Hence, there might exist a "price gap", but the "price gap" contains at most one price. Then, Proposition 2 follows by iteration.

**Lemma 4.** *In any equilibrium, $v_n^i = v_n^j = p_n$. Further, a positive measure of types of both buyers plan to buy at a price $p_n$ but not at any earlier price.*

The proof is analogous to the proof of Lemma 3 of Bose and Daripa (2009). The idea of the proof is, that all types $v \in (p_n, p_{n-1})$ do not buy at price $p \geq p_{n-1}$, since $v - p < 0$. But, since the payoff of not buying is zero, buyers of type $v \in (p_n, p_{n-1})$ prefer to buy at $p_n$.

**Lemma 5.** *In any equilibrium, a positive measure of types of each buyer plans to buy at a price $p_1$.*

Lemma 5 can be proven analogously to the proof of Lemma 4 in Bose and Daripa (2009) using the fact that $\hat{H}_k^i < 1 - \epsilon$ and that the $\epsilon$-contamination structure still holds for marginal probabilities of the rectangular hull. For completeness, we repeat it here.

*Proof.* Suppose $j$ does not plan to buy at prices $p_1, \ldots, p_k$ for $1 \leq k < n$ and $p_{k+1}$ is the first price at which $j$ buys. (This is denoted as $v_1^j = \cdots = v_k^j = 1$ and $v_{k+1}^j < 1$.) Clearly, the best response of $i$ is not to buy at prices $p_1, \ldots, p_{k-1}$. If $i$ refuses $p_k$ the probability that the game reaches $p_{k+1}$ is one. Therefore,

$$\hat{H}_k^j = \min_{g^{k+1} \in \mathrm{rect}(\Phi_j^k)} \frac{1}{2} + \frac{1}{2} g^{k+1}(v \leq v_{k+1}^i) = \frac{1}{2} + \frac{1}{2}(1 - \epsilon_{k+1}) F^{k+1}(v_{k+1}^i), \tag{B.2}$$

where the last equality follows since, $g^{k+1}(v \leq v_{k+1}^i) = G^k(v_{k+1}^i)$ and the set of marginal distributions given the rectangular hull, $\mathrm{rect}(\Phi_j^{k,\cdot})$, equals the set of marginal distributions given $\Phi_j^k$. Therefore, the $\epsilon$-contamination structure is maintained for marginal distributions of the rectangular hull.

Further, $j$ does not plan to buy at prices $p_1, \ldots, p_k$. Therefore, $v_k^j = 1$ and $F^{k+1}(v_{k+1}^i) = F(v_{k+1}^i)$ and $\epsilon_{k+1} = \epsilon$. Then, the payoff from refusing $p_k$ is $(\frac{1}{2} + \frac{1}{2}(1-\epsilon)F(v_{k+1}^j))(v - p_{k+1})$. Define the following function:

$$\hat{G}_k^i(v) := v - p_k - \left( \frac{1}{2} + \frac{1}{2}(1 - \epsilon)F(v_{k+1}^j) \right)(v - p_{k+1})$$

$$= \frac{1}{2}(v - p_k)(1 - (1 - \epsilon)F(v_{k+1}^j)) - \frac{1}{2}(1 + (1 - \epsilon)F(v_{k+1}^j))\Delta_k.$$

Note that

$$2\hat{G}_k^i(1) = (1 - p_k)(1 - (1 - \epsilon)F(v_{k+1}^j)) - (1 + (1 - \epsilon)F(v_{k+1}^j))\Delta_k$$

$$> \delta\epsilon - (2 - \epsilon)\Delta_k \geq \delta\epsilon - (2 - \epsilon)\Delta_1 = \frac{\delta\epsilon^2}{2(1 - \delta) + \delta\epsilon} > 0,$$

where the second step follows from the fact that $(1 - p_k) \geq (1 - p_1) = \delta$, and the fact that $F(v_{k+1}^j) < 1$, and the third step uses $\Delta_1 \geq \Delta_k$. Since $\hat{G}_k^i(v)$ is continuous, increasing in $v$, and negative at $v = p_k$, there exists $v_k^i$ such that $\hat{G}_k^i(v) > 0$ for $v > v_k^i$ and $\hat{G}_k^i(v_k^i) = 0$. Since we know that $i$ does not plan to buy at any earlier price than $p_k$, it must be that types $[v_k^i, 1]$ of buyer $i$ plan to buy at $p_k$. The second part shows that there exist types of $j$ close to one who would deviate and buy at $p_k$. First, remember that

$$G_k^j(v) = v - p_k - (v - p_{k+1})\hat{H}_k^j$$

with

$$\hat{H}_k^j = (1 - \epsilon_k)\frac{F^k(v_k^j) + (1 - \epsilon_{k+1})F^k(v_k^j)F^{k+1}(v_{k+1}^j)}{1 + (1 - \epsilon_k)F^k(v_k^j)} \leq (1 - \epsilon_k) \leq 1 - \epsilon.$$

Then,

$$G_k^j(1) = (1 - p_k) - (1 - p_{k+1})\hat{H}_k^j = (1 - p_k)(1 - \hat{H}_k^j) - \Delta_k\hat{H}_k^j$$

$$\geq \delta(1 - \hat{H}_k^j) - \Delta_1\hat{H}_k^j > \delta\epsilon - \Delta_1 = \delta\epsilon - \frac{\delta\epsilon}{2}\frac{1 - \delta}{(1 - \delta) + \frac{\delta\epsilon}{2}} > 0,$$

where the first inequality follows since $1 - p_k \geq 1 - p_1 = \delta$ and $\Delta_k \leq \Delta_1$. Further, $\hat{H}_k^j \leq 1 - \epsilon < 1$ implies the third inequality. The last inequality follows since $\frac{1 - \delta}{(1 - \delta) + \frac{\delta\epsilon}{2}} < 1$. Since $G_k^j(v)$ is increasing and continuous, there are types of $j$ of positive measure near one who would deviate and buy at $p_k$. Contradiction. □

**Lemma 6.** *Suppose there are types of buyer $i$ who plan to buy at $p_{n-l-t}$ and $p_{n-l}$ but not at prices in between. Then $v_{n-l-t}^i - v_n^i < \delta(l + t)$*

Similar to Lemma 5, the proof is analogously to Lemma 5 of Bose and Daripa (2009) using the fact that $\hat{H}_k^i < 1 - \epsilon$ and Equation (B.2). For more details see Lemma 9 in Section C.

Remember that we assume an equilibrium strategy where no types of $j$ plan to buy at prices in $\{p_{n-l+1}, \ldots, p_{n-1}\}$ with $l \geq 2$. Then, no types of $i$ plan to buy at prices $\{p_{n-l+1}, \ldots, p_{n-2}\}$.

**Lemma 7.** *There exists $\bar{\delta} > 0$ such that for all $\delta < \bar{\delta}$ then there are types (of positive measure) of $j$ who buy at $p_{n-1}$ if*

*i) there are types of $i$ who buy at $p_{n-1}$ or*

*ii) no types of $i$ buy at $p_{n-1}$ and $l > 2$.*

The proof of Lemma 7 explicitly uses the expression of the worst-case belief and therefore has to be adjusted.

*Proof.* In the proposed equilibrium, types $v > v_{n-l}^{j}$ of $j$ buy at prices $p \geq p_{n-l}$, with type $v_{n-l}^{j}$ and some types just above $v_{n-l}^{j}$ buying at price $p_{n-l}$. But since $j$ does not buy at prices $\{p_{n-l+1}, \ldots, p_{n-1}\}$, types just below $v_{n-l}^{j}$ must buy at $p_n$ and not before. Therefore, in the proposed equilibrium, it must be that $v_{n-l}^{j}$ is indifferent between buying at $p_{n-l}$ or $p_n$. So we have, for buyer $j$,

$$v_{n-l}^{j} - p_{n-l} = (v_{n-l}^{j} - p_n)\hat{H}_{n-l}^{j}, \tag{B.3}$$

where $\hat{H}_{n-l}^{j}$ is the worst-case belief, that $j$ gets the offer $p_n$ if he refuses $p_{n-l}$. One can derive $\hat{H}_{n-l}^{j}$ as follows. Similar to Section A, denote with $\pi_{n-l}^{j}$, the probability that the object stays unsold in period $n-l$, with $\pi_{n-1}^{j}$, the probability that the object stays unsold in period $n-1$ and with $\psi_n^{j}$ the probability that $j$ gets the offer $p_n$. Then, $\hat{H}_{n-l}^{j} = \min \pi_{n-l}^{j}\pi_{n-1}^{j}\psi_n^{j}$ with

$$\pi_{n-l}^{j} = \frac{2G^{n-l}(v_{n-l}^{i})}{1 + G^{n-l}(v_{n-l}^{i})},$$
$$\pi_{n-1}^{j} = G^{n-1}(v_{n-1}^{i}),$$
$$\psi_n^{j} = \frac{1}{2} + \frac{1}{2}G^n(v_n^{i}).$$

Similar to the proof of Proposition 1, the properties of rectangularity imply

$$\hat{H}_{n-l}^{j} = \frac{(1 - \epsilon_{n-l})F^{n-l}(v_{n-l}^{i})(1 - \epsilon_{n-1})F^{n-1}(v_{n-1}^{i})(1 + (1 - \epsilon_n)F^n(v_n^{i}))}{1 + (1 - \epsilon_{n-l})F^{n-l}(v_{n-l}^{i})}.$$

Rewriting Equation (B.3) gives,

$$v_{n-l}^{j} - p_{n-l} = \frac{(p_{n-l} - p_n)\hat{H}_{n-l}^{j}}{1 - \hat{H}_{n-l}^{j}}. \tag{B.4}$$

To establish that contrary to what has been supposed, i.e., that there are types of $j$ who will in fact buy at price $p_{n-1}$, we show that there are types $v$ such that

$$G_{n-1}^{j}(v) = v - p_{n-1} - (v - p_n)\hat{H}_{n-1}^{j} > 0,$$

where $\hat{H}_{n-1}^{j}$ is the usual rectangular worst-case belief, that $j$ gets the offer $p_n$ if he refuses $p_{n-1}$. It is useful to break up the analysis into several cases.

**Case 1)** $l$ and $t$ are fixed positive integers:
Intuitively, this is the case where both $i$ and $j$ follow strategies where they do not buy for some finite number of prices. Note that in this case, $\delta(l + t) \to 0$ as $\delta \to 0$.

i) Some types of $i$ buy at $p_{n-1}$:
In the proposed equilibrium no types of $j$ buy at $p_{n-1}$. Consider the value of $G_{n-1}^{j}(\cdot)$ at $v_{n-l}^{j}$. We have

$$G_{n-1}^{j}(v_{n-l}^{j}) = v_{n-l}^{j} - p_{n-1} - (v_{n-l}^{j} - p_n)\hat{H}_{n-1}^{j}$$

$$= (v_{n-l}^j - p_{n-l}) + (p_{n-l} - p_n) - \Delta_{n-1} - \left( (v_{n-l}^j - p_{n-l}) + (p_{n-l} - p_n) \right) \hat{H}_{n-1}^j$$

$$= (p_{n-l} - p_n) \frac{1 - \hat{H}_{n-1}^j}{1 - \hat{H}_{n-l}^j} - \Delta_{n-1}$$

$$> \left( 2 \frac{1 - \hat{H}_{n-1}^j}{1 - \hat{H}_{n-l}^j} - 1 \right) \Delta_{n-1},$$

where the second step follows from Equation (B.4) and the third step follows from the fact that $p_{n-l} - p_n \geq p_{n-2} - p_n = \Delta_{n-2} + \Delta_{n-1} > 2\Delta_{n-1}$. From the previous lemma, we know that as $\delta \to 0$: $v_{n-l-t}^i - v_n^i \to 0$, $v_{n-l}^i - v_n^i < v_{n-l-t}^i - v_n^i \to 0$ and $v_{n-1}^i - v_n^i < v_{n-l-t}^i - v_n^i \to 0$. Then, using

$$\epsilon_{n-l} = \frac{\epsilon}{(1 - \epsilon)F(v_{n-l-t}^i)) + \epsilon}, \qquad\qquad F^{n-l} = \frac{F(v_{n-l}^i)}{F(v_{n-l-t}^i)},$$

$$\epsilon_{n-1} = \frac{\epsilon}{(1 - \epsilon)F(v_{n-l}^i)) + \epsilon}, \qquad\qquad F^{n-1} = \frac{F(v_{n-1}^i)}{F(v_{n-l}^i)},$$

$$\epsilon_n = \frac{\epsilon}{(1 - \epsilon)F(v_{n-1}^i)) + \epsilon}, \qquad\qquad F^n = \frac{F(v_n^i)}{F(v_{n-1}^i)},$$

and Lemma 6, one can show that

$$\lim_{\delta \to 0} \frac{1 - \hat{H}_{n-1}^j}{1 - \hat{H}_{n-l}^j} > \frac{1}{2}.$$

Hence for sufficiently small $\delta$, we have $G_{n-1}^j(v_{n-l}^j) > 0$.

ii) No types of $i$ buy at $p_{n-1}$ and $l > 2$:

In this case, if buyer $j$ refuses $p_{n-1}$, he knows that the game proceeds to the next stage. Then, analogously to Equation (B.2) we get

$$\hat{G}_{n-1}^j(v) = v - p_{n-1} - \left( \frac{1}{2} + \frac{1}{2}(1 - \epsilon_n)F^n(v_n^i) \right) (v - p_n).$$

It follows that[7]

$$2\hat{G}_{n-1}^j(v_{n-l}^j) = (v_{n-l}^j - p_{n-1}) \left( 1 - (1 - \epsilon_n)F^n(v_n^i) \right) - \Delta_{n-1} \left( 1 + (1 - \epsilon_n)F^n(v_n^i) \right)$$

$$= (v_{n-l}^j - p_{n-l} + p_{n-l} - p_n + p_n - p_{n-1}) \left( 1 - (1 - \epsilon_n)F^n(v_n^i) \right)$$
$$\qquad - \Delta_{n-1} \left( 1 + (1 - \epsilon_n)F^n(v_n^i) \right)$$

$$= (v_{n-l}^j - p_{n-l}) \left( 1 - (1 - \epsilon_n)F^n(v_n^i) \right)$$
$$\qquad + (p_{n-l} - p_n) \left( 1 - (1 - \epsilon_n)F^n(v_n^i) \right) - 2\Delta_{n-1}$$

$$= (p_{n-l} - p_n) \left( 1 - (1 - \epsilon_n)F^n(v_n^i) \right) \left( \frac{\hat{H}_{n-l}^j}{1 - \hat{H}_{n-l}^j} + 1 \right) - 2\Delta_{n-1}$$

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[7]Please note that, due to a calculation error in Bose and Daripa (2009), our expression slightly deviates from their expression.

$$= (p_{n-l} - p_n)\frac{1 - (1 - \epsilon_n)F^n(v_n^i)}{1 - \hat{H}_{n-l}^j} - 2\Delta_{n-1}$$

$$> \left(l\frac{1 - (1 - \epsilon_n)F^n(v_n^i)}{1 - \hat{H}_{n-l}^j} - 2\right)\Delta_{n-1},$$

where the fourth step follows from Equation (B.4) and the final inequality follows, as before, from the fact that $\Delta_k$ is decreasing in $k$ and $p_{n-l} - p_n = p_{n-l} - p_{n-l+1} + p_{n-l+1} - \cdots - p_n > l\Delta_{n-1}$. Similar to Case 1.1) one can show that as $\delta \to 0$

$$\frac{1 - (1 - \epsilon_n)F^n(v_n^i)}{1 - \hat{H}_{n-l}^j} \to 1$$

and we have for $\delta$ sufficiently small $\hat{G}_{n-1}^j(v_{n-l}^j) > 0$ since $l > 2$.[8]

Analogously to Bose and Daripa (2009), we can show that in Case 2) ($t$ is arbitrary and $l$ varies with $n$) and Case 3) ($l$ is a fixed integer and $t$ varies with $n$) $G_{n-1}^j(v_{n-l}^j) > 0$ and $\hat{G}_{n-1}^j(v_{n-l}^j) > 0$. For completeness, we repeat it here:

**Case 2)** $t$ is arbitrary and $l$ varies with $n$:

In this case the gap $p_{n-l} - p_{n-1}$ does not vanish as $\delta \to 0$. Then, since for any given $\eta > 0$, $\hat{H}_{n-l}^j$ is bounded away from zero, it follows from that $v_{n-l}^j - p_{n-l}$ does not vanish. Since $p_{n-1} < p_{n-l}$, also $v_{n-l}^j - p_{n-1}$ does not vanish. However, $p_n - p_{n-1} \to 0$ as $\delta \to 0$ and $\hat{H}_{n-1}^j < 1$ (for Case 1.i)) and $(1 - \epsilon_n F^n(v_n^i) < 1$ (for Case 1.ii)). Therefore, for $\delta$ small enough, $G_{n-1}^j(v_{n-l}^j) > 0$ and $\hat{G}_{n-1}^j(v_{n-l}^j) > 0$.

**Case 3)** $l$ is a fixed integer and $t$ varies with $n$:

This is the case when as $\delta \to 0$, $\delta(l+t)$ does not go to zero because $t$ (and $n$) become arbitrarily large as $\delta$ becomes small. However, this is analogous to the case we have analyzed before with $i$ and $j$ roles switched. We know that in equilibrium, both buyers have types who plan to buy at the price $p_{n-l}$. If $i$ plans to buy at prices $p_{n-l-t}$ and $p_{n-l}$, but not to buy in between, the best response of $j$ should involve not buying at prices $\{p_{n-l-t+1}, \ldots, p_{n-k-2}\}$. If $p_{n-l-t} - p_{n-l-1}$ does not go to zero, we can use the arguments from Case 2) above to argue that contrary to what is being supposed, for small $\delta$, buyer $i$ will have some types of positive measure who buy at $p_{n-l-1}$ rather than waiting till $p_{n-l}$

Now, since $G_{n-1}^j(\cdot)$ (and $\hat{G}_{n-1}^j(\cdot) > 0$) is strictly increasing, continuous, and negative at $p_{n-1}$, there is $v_{n-1}^j \in (p_{n-1}, v_{n-l}^j)$ such that $G_{n-1}^j(v) > 0$ (and $\hat{G}_{n-1}^j(v) > 0$ ) for $v \in (v_{n-1}^j, v_{n-l}^j)$. Since types below $v_{n-l}^j$ do not buy at any price greater than or equal to $p_{n-l}$, these types (of positive measure) strictly prefer to stop at $p_{n-1}$ rather than wait till $p_n$. This contradicts the supposed equilibrium. □

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[8]The condition $l > 2$ is needed, since without further conditions on $F$ and $\epsilon$ it is not clear if $\frac{1-(1-\epsilon_n)F^n(v_n^i)}{1-\hat{H}_{n-l}^j} \nearrow 1$ or $\frac{1-(1-\epsilon_n)F^n(v_n^i)}{1-\hat{H}_{n-l}^j} \searrow 1$. Therefore, for $l = 2$ we get $\hat{G}_{n-1}^j(v_{n-l}^j) \geq 0$ and it could be possible, that $j$ is indifferent between buying at $p_{n-2}$ or buying at $p_{n-1}$ or waiting till $p_n$.

# C Correction Results of Bose and Daripa (2009)

## C.1 Proof of Lemma 3

We first derive the worst-case belief $\bar{H}_k^i = \min_{g \in \Phi_i^k} \mathbb{P}^{g^k}(A|B)$. Then, we show that $\bar{H}_k^i$ differs from $(1 - \epsilon)H_k^i$. For an arbitrary fixed $g^k \in \Phi_i^k$ one can calculate $\mathbb{P}^{g^k}(A|B)$ analogously to $H_k^i$ in Appendix A.1. of Bose and Daripa (2009).[9] Using $g^k(\cdot) = \frac{g(\cdot)}{g(v < v_{k+1}^j)}$, we can rewrite $\min_{g \in \Phi_i^k} \mathbb{P}^{g^k}(A|B)$ to

$$\min_{g \in \Phi_i^{-1}} \mathbb{P}^g(A|B) = \min_{g \in \Phi_i^{-1}} \frac{G(v_k^j) + G(v_{k+1}^j)}{G(v_k^j) + G(v_{k-1}^j)},$$

where $G$ denotes the distribution function of $g$. The $\epsilon$-contamination structure of $G$ gives

$$\mathbb{P}^g(A|B) = \frac{(1-\epsilon)(F(v_k^j) + F(v_{k+1}^j)) + \epsilon(L(v_k^j) + L(v_{k+1}^j))}{(1-\epsilon)(F(v_k^j) + F(v_{k-1}^j)) + \epsilon(L(v_k^j) + L(v_{k-1}^j))}. \tag{C.1}$$

Then, a worst-case belief $g^* = \arg\min_{g \in \Phi_i^{-1}} \mathbb{P}^g(A|B)$ has to satisfy $L^*(v_{k+1}^j) = L^*(v_k^j) = 0 < 1 = L^*(v_{k-1}^j)$, which is well defined since $v_{k-1}^j \geq v_k^j \geq v_{k+1}^j$. $L^*(v_{k+1}^j) = 0$ and $L^*(v_{k-1}^j) = 1$ follows immediately from Equation (C.1). Furthermore, $v_{k-1}^j \geq v_{k+1}^j$ implies that $\mathbb{P}^g(A|B)$ is monotone increasing in $L^*(v_k^j)$. Therefore, $L^*(v_k^j) = 0$. Then,

$$\bar{H}_k^i = \mathbb{P}^{g^*}(A|B) = \frac{(1-\epsilon)(F(v_k^j) + F(v_{k+1}^j))}{(1-\epsilon)(F(v_k^j) + F(v_{k-1}^j)) + \epsilon}.$$

$\bar{H}_k^i$ is smaller than $(1-\epsilon)H_k^i$ if $F(v_k^j) + F(v_{k-1}^j) < 1$ and greater than $(1-\epsilon)H_k^i$ if $F(v_k^j) + F(v_{k-1}^j) > 1$. Hence, $\bar{H}_k^i = (1-\epsilon)H_k^i$ if and only if $F(v_k^j) + F(v_{k-1}^j) = 1$. But, this condition is cannot be satisfied for all $k = 1, \ldots, n$.

## C.2 Correction further Results of Bose and Daripa (2009)

Using $\bar{H}_k^i$ instead of $(1-\epsilon)H_k^i$ leads to several problems in the proofs of Bose and Daripa (2009). Many proofs use the fact that $(1-\epsilon)H_k^i < 1 - \epsilon$, but this does not hold for $\bar{H}_k^i$. More precisely, there exists $f$ with full support on $[0,1]$ and $\mathbf{v}^j$, such that $\bar{H}_k^i > (1-\epsilon)$. The numerical example in Section 5.2 illustrates this problem.

We now summarize the results of Bose and Daripa (2009) and show how the proofs must be adapted when using the corrected worst-case belief.

**Proposition 1 of Bose and Daripa (2009)** Bose and Daripa (2009) first show that there exists $\bar{\delta} > 0$ such that for all $\delta < \bar{\delta}$, the equilibrium strategy of both players are interior cut-off strategy. The formal proof consists of several lemmata and uses the fact that $H_k^i < 1$ frequently. However, the result still holds under the corrected worst-case belief. Even if the idea of the proof stays the same, many technical details must be adjusted. First, as in Bose and Daripa (2009), one can show that there exists a positive measure of types of both buyers who plan to buy at a price $p_n$ but not at any earlier price.

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[9]For completeness we derive $\mathbb{P}^{g^k}(A|B)$ in Section A in the Appendix.

**Lemma 8** (Lemma 4 of Bose and Daripa (2009)). *In any equilibrium, a positive measure of types of each buyer plan to buy at a price $p_1$.*

*Proof.* The first part of the proof still holds given the corrected worst-case belief. Suppose $j$ does not plan to buy at prices $p_1, \ldots, p_k$ for $1 \leq k < n$ and $p_{k+1}$ is the first price at which $j$ buys. (This is denoted as $v_1^j = \cdots = v_k^j = 1$ and $v_{k+1}^j < 1$.) Clearly, the best response of $i$ is not to buy at prices $p_1, \ldots, p_{k-1}$. If $i$ refuses $p_k$ the probability that the game reaches $p_{k+1}$ is one, i.e., $\pi_k^i = 1$. Therefore,

$$\bar{H}_k^i = \min_{g \in \Phi_i^k} \frac{1}{2} + \frac{1}{2} G^{k+1}(v_{k+1}^j)$$

$$= \frac{1}{2} + \frac{1}{2}(1 - \epsilon_{k+1}) F^{k+1}(v_{k+1}^j) = \frac{1}{2} + \frac{1}{2}(1 - \epsilon) F(v_{k+1}^j), \tag{C.2}$$

where the last step follows from $v_k^j = 1$. Therefore, the payoff from refusing $p_k$ is $(\frac{1}{2} + \frac{1}{2}(1 - \epsilon) F(v_{k+1}^j))(v - p_{k+1})$. Define the following function:

$$\hat{G}_k^i(v) := v - p_k - \left( \frac{1}{2} + \frac{1}{2}(1 - \epsilon) F(v_{k+1}^j) \right)(v - p_{k+1}) \tag{C.3}$$

$$= \frac{1}{2}(v - p_k)(1 - (1 - \epsilon) F(v_{k+1}^j)) - \frac{1}{2}(1 + (1 - \epsilon) F(v_{k+1}^j)) \Delta_k.$$

Note that

$$2\hat{G}_k^i(1) = (1 - p_k)(1 - (1 - \epsilon) F(v_{k+1}^j)) - (1 + (1 - \epsilon) F(v_{k+1}^j)) \Delta_k$$

$$> \delta\epsilon - (2 - \epsilon)\Delta_k \geq \delta\epsilon - (2 - \epsilon)\Delta_1 = \frac{\delta\epsilon^2}{2(1 - \delta) + \delta\epsilon} > 0,$$

where the second step follows from the fact that $(1 - p_k) \geq (1 - p_1) = \delta$, and the fact that $F(v_{k+1}^j) < 1$, and the third step uses $\Delta_1 \geq \Delta_k$. Since $\hat{G}_k^i(v)$ is continuous, increasing in $v$, and negative at $v = p_k$, there exists $v_k^i$ such that $\hat{G}_k^i(v) > 0$ for $v > v_k^i$ and $\hat{G}_k^i(v_k^i) = 0$. Since we know that $i$ does not plan to buy at any earlier price than $p_k$, it must be that types $[v_k^i, 1]$ of buyer $i$ plan to buy at $p_k$. For the second part, Bose and Daripa (2009) use the fact that $(1 - \epsilon) H_k^i < 1 - \epsilon$. However, this does not hold given $\bar{H}_k^i$. Therefore, we have to adjust the second part. Using the definition of $\bar{H}_k^i$, we can still show, that $G_k^j(1) > 0$. First, remember that

$$G_k^j(v) = v - p_k - (v - p_{k+1}) \bar{H}_k^j$$

with the worst-case belief

$$\bar{H}_k^j = \frac{(1 - \epsilon)(F(v_k^i) + F(v_{k+1}^i))}{(1 - \epsilon)(F(v_k^i) + F(v_{k-1}^i)) + \epsilon}.$$

Then,

$$G_k^j(1) = (1 - p_k) - (1 - p_{k+1}) \bar{H}_k^j$$

$$= (1 - p_k)(1 - \bar{H}_k^j) - \Delta_k \bar{H}_k^j$$

$$\geq \delta(1 - \bar{H}_k^j) - \Delta_1 \bar{H}_k^j$$

$$= \delta \frac{(1-\epsilon)(F(v^i_{k-1}) - F(v^i_{k+1})) + \epsilon}{(1-\epsilon)(F(v^i_k) + F(v^i_{k-1})) + \epsilon} - \frac{\delta\epsilon}{2} \frac{1-\delta}{(1-\delta) + \frac{\delta\epsilon}{2}} \frac{(1-\epsilon)(F(v^i_k) + F(v^i_{k+1}))}{(1-\epsilon)(F(v^i_k) + F(v^i_{k-1})) + \epsilon}$$

$$\geq \delta \frac{(1-\epsilon)(F(v^i_{k-1}) - F(v^i_{k+1})) + \epsilon - \epsilon(1-\epsilon)}{(1-\epsilon)(F(v^i_k) + F(v^i_{k-1})) + \epsilon} > 0,$$

where the first inequality follows since $1 - p_k \geq 1 - p_1 = \delta$ and $\Delta_k \leq \Delta_1$. The second inequality follows since $\frac{1-\delta}{(1-\delta)+\frac{\delta\epsilon}{2}} < 1$ and $(F(v^i_k) + F(v^i_{k+1})) \leq 2$ and the last inequality since $(F(v^i_{k-1}) - F(v^i_{k+1})) \geq 0$. Since $G^j_k(v)$ is increasing and continuous, there are types of $j$ of positive measure near one who would deviate and buy at $p_k$. Contradiction. $\square$

**Lemma 9** (Lemma 5 of Bose and Daripa (2009)). *Suppose there are types of buyer $i$ who plan to buy at $p_{n-l-t}$ and $p_{n-l}$ but not at prices in between. Then $v^i_{n-l-t} - v^i_n < \delta(l+t)$*

*Proof.* Since there are no types who buy at prices between $p_{n-l-t}$ and $p_{n-l}$, buyer $j$ will never accept a price in $\{p_{n-l-t+1}, \ldots, p_{n-l-2}\}$.
**Case 1)** Some types of buyer $j$ buy at prices $p_{n-l-t}$ and/or $p_{n-l-1}$, $t \geq 1$:
In this case, if $i$ refuses $p_{n-l-t}$ it is possible, that the game ends before, $p_{n-l}$ is offered. We know, that $v^i_{n-l-t}$ is given by $G^i_{n-l-t}(v) = 0$, i.e.,

$$v^i_{n-l-t} - p_{n-l-t} = (v^i_{n-l-t} - p_{n-l})\bar{H}^i_{n-l-t}$$
$$= (v^i_{n-l-t} - p_{n-l-t} + \Delta_{n-l-t} + \cdots + \Delta_{n-l-1})\bar{H}^i_{n-l-t}$$
$$\Leftrightarrow \quad v^i_{n-l-t} - p_{n-l-t} = (\Delta_{n-l-t} + \cdots + \Delta_{n-l-1})\frac{\bar{H}^i_{n-l-t}}{1 - \bar{H}^i_{n-l-t}}. \tag{C.4}$$

Here $\bar{H}^i_{n-l-t}$ is the worst-case probability that $i$ gets the offer $p_{n-l}$ if he rejects the current price $p_{n-l-t}$. Since there are types of buyer $i$, who plan to buy at $p_{n-l-1}$, $\bar{H}^i_{n-l-t}$ can be derived from the following three probabilities. Let $\pi^i_{n-l-t}$ denote the probability that the object remains unsold at $p_{n-l-t}$, $\hat{\pi}^i_{n-l-1}$ denotes the probability that the object remains unsold at $p_{n-l-1}$ and $\psi^i_{n-l}$ the probability that $i$ gets the offer $p_{n-l}$. Similar to Section A, one can show that

$$\pi^i_{n-l-t} = \frac{2G(v^j_{n-l-t})}{G(v^j_{n-l-1}) + G(v^j_{n-l-t})},$$
$$\hat{\pi}^i_{n-l-1} = \frac{G(v^j_{n-l-1})}{G(v^j_{n-l-t})},$$
$$\psi^i_{n-l} = \frac{1}{2} + \frac{1}{2}\frac{G(v^j_{n-l})}{G(v^j_{n-l-1})}. \tag{C.5}$$

The worst-case probability is then given by

$$\bar{H}^i_{n-l-t} = \min_{g \in \Phi^{-1}_i} \pi^i_{n-l-t}\hat{\pi}^i_{n-l-1}\psi^i_{n-l} = \min_{g \in \Phi^{-1}_i} \frac{G(v^j_{n-l-1}) + G(v^j_{n-l})}{G(v^j_{n-l-t-1}) + G(v^j_{n-l-t})}$$
$$= \frac{(1-\epsilon)(F(v^j_{n-l-1}) + F(v^j_{n-l}))}{(1-\epsilon)(F(v^j_{n-l-t-1}) + F(v^j_{n-l-t})) + 2\epsilon}. \tag{C.6}$$

Further,

$$\frac{\bar{H}^i_{n-l-t}}{1 - \bar{H}^i_{n-l-t}} = \frac{(1-\epsilon)(F(v^j_{n-l-1}) + F(v^j_{n-l}))}{(1-\epsilon)(F(v^j_{n-l-t-1}) + F(v^j_{n-l-t}) - F(v^j_{n-l-1}) - F(v^j_{n-l})) + 2\epsilon}$$

$$\leq \frac{2(1-\epsilon)}{2\epsilon} = \frac{1-\epsilon}{\epsilon}, \tag{C.7}$$

where the inequality holds since, $F(v^j_{n-l-1}) + F(v^j_{n-l}) \leq 2$ and $F(v^j_{n-l-t-1}) + F(v^j_{n-l-t}) - F(v^j_{n-l-1}) - F(v^j_{n-l}) \geq 0$. The rest of the proof follows as in Bose and Daripa (2009). We repeat it for completeness. Let $\alpha := \frac{1-\delta}{1-\delta+\frac{\delta\epsilon}{2}}$. Note, that $\alpha > 1$ and from the definition of $\Delta_k$, we have $\Delta_k = \frac{1}{2}\delta\epsilon\alpha^k < \frac{1}{2}\delta\epsilon$. Therefore,

$$v^i_{n-l-t} - p_n = v^i_{n-l-t} - p_{n-l-t} + \Delta_{n-l-t} + \cdots + \Delta_{n-1}$$

$$= (\Delta_{n-l-t} + \cdots + \Delta_{n-1})\frac{\bar{H}^i_{n-l-t}}{1 - \bar{H}^i_{n-l-t}} + \Delta_{n-l-t} + \cdots + \Delta_{n-1}$$

$$\leq (\Delta_{n-l-t} + \cdots + \Delta_{n-1})\frac{1-\epsilon}{\epsilon} + \Delta_{n-l-t} + \cdots + \Delta_{n-1}$$

$$= (\Delta_{n-l-t} + \cdots + \Delta_{n-l-2})\frac{1-\epsilon}{\epsilon} + \frac{\Delta_{n-l-1}}{\epsilon} + \Delta_{n-l} + \cdots + \Delta_{n-1}$$

$$< \frac{1}{2}\left(\delta(1-\epsilon)(t-1) + \delta + \delta\epsilon l\right) < \delta(t + \epsilon(1-t) + \epsilon l) < \delta(l+t),$$

where the second step follows from Equation (C.4), the third step from Equation (C.7) and the fifth step uses $\Delta_k < \frac{1}{2}\delta\epsilon$ for all $k = n-l-t, \ldots, n-1$. Finally, since $v^i_n = p_n$, $v^i_{n-l-t} - v^i_n < \delta(l+t)$.[10]

The proof of Case 2), follows as in Bose in Daripa. For completeness, we repeat it here.

**Case 2)** No types of buyer $j$ buy at prices $p_{n-l-t}$ and $p_{n-l-1}$, $t \geq 1$:

We know that types of $i$ buy at $p_{n-l-t}$ and at $p_{n-l}$ but not at the prices in between. If $t > 1$, any type of $i$ who buys at $p_{n-l-t}$ can deviate profitably and buy at $p_{n-l-1}$ instead. Contradiction. Therefore in this case the only possibility is $t = 1$.

So it remains to prove the inequality when $t = 1$ and no type of $j$ buys at $p_{n-l-1}$. In this case, analogously to Equation (C.3) $v^i_{n-l-1}$ is given by

$$\hat{G}^i_{n-l-1}(v) = v - p_{n-l-1} - \left(\frac{1}{2} + \frac{1}{2}R^i_{n-l-1}\right)(v - p_{n-l}) = 0,$$

where $R^i_{n-l-1}$ is the conditional probability that $j$ rejects $p_{n-l}$.[11] Using the fact, that $v - p_{n-l} = v - p_{n-l-1} + \Delta_{n-l-1}$, and solving

$$v^i_{n-l-1} - p_{n-l} = \Delta_{n-l-1}\frac{1 + R^i_{n-l-1}}{1 - R^i_{n-l-1}} < \Delta_{n-l-1}\frac{2-\epsilon}{\epsilon},$$

---

[10]For $t = 1$, or if $j$ only buys at $p_{n-l-t}$ OR $p_{n-l-1}$, $v^j_{n-l-t} = v^j_{n-l-1}$ and the worst-case belief changes slightly. However, similar to above it follows that $\frac{\bar{H}^i_{n-l-t}}{1 - \bar{H}^i_{n-l-t}} \leq \frac{2(1-\epsilon)}{\epsilon}$ and $v^i_{n-l-t} - p_{n-l-t} \leq \delta(1-\epsilon)(t-1) + \delta + \delta\epsilon l < \delta(l+t)$.

[11]Suppose the lowest price higher than $p_{n-l}$ at which some types of $j$ buy is $p_{n-l-1-s}$. Then $R^i_{n-l-1} = (1 - \epsilon_{n-l-1})\frac{F(v^j_{n-l})}{F(v^j_{n-l-1-s})}$.

where the inequality uses the fact that $1 - \epsilon_n < 1 - \epsilon$. Similar to Case 1), it follows

$$
\begin{aligned}
v^i_{n-l-1} - p_n &= v^i_{n-l-1} - p_{n-l-1} + \Delta_{n-l-1} + \cdots + \Delta_{n-1} \\
&< \Delta_{n-l-1}\left(\frac{2-\epsilon}{\epsilon} + 1\right) + \Delta_{n-l} + \cdots + \Delta_{n-1} \\
&= \Delta_{n-l-1}\frac{2}{\epsilon} + \Delta_{n-l} + \cdots + \Delta_{n-1} < \delta + \frac{\epsilon\delta}{2}l < \delta(l+1).
\end{aligned}
$$

$\square$

Suppose in equilibrium no types of $j$ plan to buy at prices in $\{p_{n-l+1}, \ldots, p_{n-1}\}$. Then, there are no types of $i$ who plan to buy at prices $\{p_{n-l+1}, \ldots, p_{n-2}\}$. The next lemma shows that given this equilibrium strategy, there are types of buyer $j$ who can profit from deviating and buy at $p_{n-1}$. Hence, the strategy described above cannot be an equilibrium. Then, Proposition 1 follows from iterating this result.

**Lemma 10** (Lemma 6 of Bose and Daripa (2009)). *There is $\bar{\delta} > 0$ such that for all $\delta < \bar{\delta}$ there are types (of positive measure) of $j$ who buy at $p_{n-1}$.*

*Proof.* Again, the structure is similar to Bose and Daripa (2009). Case 2) and 3) follow analogously to Bose and Daripa (2009) and the proof of Lemma 7. Only Case 1) ($l$ and $t$ are fixed positive integers) requires some adjustments In the proposed equilibrium, types $v > v^j_{n-l}$ of $j$ buy at prices $p \geq p_{n-l}$, with type $v^j_{n-l}$ and some types just above buying at price $p_{n-l}$. But since $j$ does not buy at prices $\{p_{n-l+1}, \ldots, p_{n-1}\}$, types just below $v^j_{n-l}$ must buy at $p_n$ and not before. Therefore, in the proposed equilibrium, it must be that $v^j_{n-l}$ is indifferent between buying at $p_{n-l}$ or $p_n$. So we have, for buyer $j$,

$$
v^j_{n-l} - p_{n-l} = (v^j_{n-l} - p_n)\bar{H}^j_{n-l}, \tag{C.8}
$$

where $\bar{H}^j_{n-l}$ is the worst-case belief, that $j$ gets the offer $p_n$ if he refuses $p_{n-l}$. One can derive $\bar{H}^j_{n-l}$ similar to Equation (C.6)

$$
\bar{H}^j_{n-l} = \frac{(1-\epsilon)(F(v^i_{n-1} + F(v^i_n))}{(1-\epsilon)(F(v^i_{n-l} + F(v^i_{n-l-t})) + 2\epsilon}
$$

if there are types of $i$ who buy at $p_{n-1}$. If no types of $i$ plan to buy at $p_{n-1}$, then $v^i_{n-1} = v^i_{n-l}$ and $2\epsilon$ in the denominator has to be replaced by $\epsilon$.

Rewriting Equation (C.8) gives,

$$
v^j_{n-l} - p_{n-l} = \frac{(p_{n-l} - p_n)\bar{H}^j_{n-l}}{1 - \bar{H}^j_{n-l}}. \tag{C.9}
$$

To establish that there are types of $j$ who will in fact want to buy at price $p_{n-1}$, we show that there are types $v$ such that

$$
G^j_{n-1}(v) = v - p_{n-1} - (v - p_n)\bar{H}^j_{n-1} > 0,
$$

where $\bar{H}^j_{n-1}$ is the usual worst-case belief, that $j$ gets the offer $p_n$ if he refuses $p_{n-1}$.

Case 1.1) Some types of $i$ buy at $p_{n-1}$:

In the proposed equilibrium, no types of $j$ buy at $p_{n-1}$. Therefore, it must be that $G_{n-1}^{j}(v)$ is not strictly positive for any $v \in [p_{n-1}, v_{n-l}^{j}]$. Consider the value of $G_{n-1}^{j}(\cdot)$ at $v_{n-l}^{j}$. We have

$$
\begin{aligned}
G_{n-1}^{j}(v_{n-l}^{j}) &= v_{n-l}^{j} - p_{n-1} - (v_{n-l}^{j} - p_n)\bar{H}_{n-1}^{j} \\
&= (v_{n-l}^{j} - p_{n-l}) + (p_{n-l} - p_n) - \Delta_{n-1} - \left((v_{n-l}^{j} - p_{n-l}) + (p_{n-l} - p_n)\right)\bar{H}_{n-1}^{j} \\
&= (p_{n-l} - p_n)\frac{1 - \bar{H}_{n-1}^{j}}{1 - \bar{H}_{n-l}^{j}} - \Delta_{n-1} \\
&> \left(2\frac{1 - \bar{H}_{n-1}^{j}}{1 - \bar{H}_{n-l}^{j}} - 1\right)\Delta_{n-1},
\end{aligned}
$$

where the third step follows from the fact that $p_{n-l} - p_n \geq p_{n-2} - p_n = \Delta_{n-2} + \Delta_{n-1} > 2\Delta_{n-1}$. Then,

$$
\begin{aligned}
&\frac{1 - \bar{H}_{n-1}^{j}}{1 - \bar{H}_{n-l}^{j}} \\
&= \frac{\left((1-\epsilon)(F(v_{n-l}^{i}) - F(v_n^{j})) + \epsilon\right)\left((1-\epsilon)(F(v_{n-l}^{i}) + F(v_{n-l+t}^{j})) + 2\epsilon\right)}{\left((1-\epsilon)(F(v_{n-l}^{i}) + F(v_{n-l+t}^{j}) - F(v_{n-1}^{i}) - F(v_n^{i})) + 2\epsilon\right)} \\
&\qquad \cdot \frac{1}{\left((1-\epsilon)(F(v_{n-l}^{i})F(v_{n-1}^{j})) + \epsilon\right)}.
\end{aligned}
$$

From the previous lemma, we know that as $\delta \to 0$: $v_{n-l-t}^{i} - v_n^{i} \to 0$, $v_{n-l}^{i} - v_n^{i} < v_{n-l-t}^{i} - v_n^{i} \to 0$ and $v_{n-1}^{i} - v_n^{i} < v_{n-l-t}^{i} - v_n^{i} \to 0$. Therefore, as $\delta \to 0$

$$
\frac{1 - \bar{H}_{n-1}^{j}}{1 - \bar{H}_{n-l}^{j}} \to \frac{\epsilon(1-\epsilon)F(v_n^{i}) + \epsilon^2}{2\epsilon(1-\epsilon)F(v_n^{i}) + \epsilon^2} > \frac{\epsilon(1-\epsilon)F(v_n^{i}) + \epsilon^2}{2\left(\epsilon(1-\epsilon)F(v_n^{i}) + \epsilon^2\right)} = \frac{1}{2}.
$$

Hence for sufficiently small $\delta$, we have $G_{n-1}^{j}(v_{n-l}^{j}) > 0$.

Case 1.2) No types of $i$ buy at $p_{n-1}$:

In this case, if buyer $j$ refuses $p_{n-1}$, he knows that the game proceeds to the next stage, and with probability $\frac{1}{2}$, he gets the offer next period. Then,

$$
\hat{G}_{n-1}^{j}(v) = v - p_{n-1} - \left(\frac{1}{2} + \frac{1}{2}(1-\epsilon_n)F^n(v_n^{i})\right)(v - p_n).
$$

It follows that[12]

$$
\begin{aligned}
2\hat{G}_{n-1}^{j}(v_{n-l}^{j}) &= (v_{n-l}^{j} - p_{n-1})\left(1 - (1-\epsilon_n)F^n(v_n^{i})\right) - \Delta_{n-1}\left(1 + (1-\epsilon_n)F^n(v_n^{i})\right) \\
&= (v_{n-l}^{j} - p_{n-l} + p_{n-l} - p_n + p_n - p_{n-1})\left(1 - (1-\epsilon_n)F^n(v_n^{i})\right) \\
&\quad - \Delta_{n-1}\left(1 + (1-\epsilon_n)F^n(v_n^{i})\right)
\end{aligned}
$$

---

[12]Please note that, due to a calculation error in Bose and Daripa (2009), our expression slightly deviates from their expression.

$$= (v_{n-l}^j - p_{n-l})\left(1 - (1 - \epsilon_n)F^n(v_n^i)\right)$$
$$+ (p_{n-l} - p_n)\left(1 - (1 - \epsilon_n)F^n(v_n^i)\right) - 2\Delta_{n-1}$$
$$= (p_{n-l} - p_n)\left(1 - (1 - \epsilon_n)F^n(v_n^i)\right)\left(\frac{\bar{H}_{n-l}^j}{1 - \bar{H}_{n-l}^j} + 1\right) - 2\Delta_{n-1}$$
$$= (p_{n-l} - p_n)\frac{1 - (1 - \epsilon_n)F^n(v_n^i)}{1 - \bar{H}_{n-l}^j} - 2\Delta_{n-1}$$
$$> \left(2\frac{1 - (1 - \epsilon_n)F^n(v_n^i)}{1 - \bar{H}_{n-l}^j} - 2\right)\Delta_{n-1}.$$

Now, similar to Case 1.1) we know that as $\delta \to 0$: $v_{n-l-t}^i - v_n^i \to 0$, $v_{n-l}^i - v_n^i < v_{n-l-t}^i - v_n^i \to 0$ and $v_{n-1}^i - v_n^i < v_{n-l-t}^i - v_n^i \to 0$. Therefore, as $\delta \to 0$

$$\frac{1 - (1 - \epsilon_n)F^n(v_n^i)}{1 - \bar{H}_{n-l}^j} \to \frac{2(1 - \epsilon)F(v_n^i) + \epsilon}{(1 - \epsilon)F(v_n^i) + \epsilon} > 1.$$

Hence, for sufficiently small $\delta$, $\hat{G}_{n-1}^j(v_{n-l}^j) > 0$.

Analogously to Bose and Daripa (2009), we can show that in Case 2) ($t$ is arbitrary and $l$ varies with $n$) and Case 3) ($l$ is a fixed integer and $t$ varies with $n$) $G_{n-1}^j(v_{n-l}^j) > 0$ and $\hat{G}_{n-1}^j(v_{n-l}^j) > 0$. In the proof of Lemma 7, we explain Cases 2) and 3) in more detail.

Now, since $G_{n-1}^j(\cdot)$ (and $\hat{G}_{n-1}^j(\cdot)$) is strictly increasing, continuous, and negative at $p_{n-1}$, there is $v_{n-1}^j \in (p_{n-1}, v_{n-l}^j)$ such that $G_{n-1}^j(v) > 0$ (and $\hat{G}_{n-1}^j(v) > 0$ ) for $v \in (v_{n-1}^j, v_{n-l}^j)$. Since types below $v_{n-l}^j$ do not buy at any price greater than or equal to $p_{n-l}$, these types (of positive measure) strictly prefer to stop at $p_{n-1}$ rather than wait till $p_n$. This contradicts the supposed equilibrium. $\square$

**Lemma 2 of Bose and Daripa (2009)**  Lemma 2 of Bose and Daripa (2009) generalize the cut-off strategies to perfect cut-off strategies. If $p_k(v)$ is the highest price that type $v$ would accept. Then, monotonicity implies that $v$ accepts all prices lower than $p_k(v)$. The result and the proof do not depend on $\bar{H}_k^i$ and, therefore, still hold with the corrected worst-case belief.

**Proposition 2 of Bose and Daripa (2009)**  Next, Bose and Daripa (2009) characterize the cut-off types in an equilibrium. Using our corrected worst-case belief, the notation of the statement changes slightly. However, the intuition and the proof follow analogously to the proof of Bose and Daripa (2009).

**Proposition 5** (Proposition 2 of Bose and Daripa (2009)). *For $\delta < \bar{\delta}$, in any equilibrium the strategy of any buyer $i$ is a perfect cut-off strategy $\mathbf{v}^i = (v_i^i, \ldots, v_n^i)$, where $v_n^i = p_n$. Further, for $1 \le k \le n-1$, $v_k^i \in (p_k, v_{k-1}^i)$, where $v^i = 0$ and $v_k^i$ is given by*

$$v_k^i = p_k + \Delta_k \frac{\bar{H}_k^i}{1 - \bar{H}_k^i},$$

*where $\bar{H}_k^i$ is the corrected worst-case belief. Further, for any given $\mathbf{v}^j$, $v_k^i$ is unique.*

**Proposition 3 of Bose and Daripa (2009)** Proposition 3 of Bose and Daripa (2009) shows the existence of a symmetric equilibrium for all $\delta < \underline{\delta}$ with $\underline{\delta} > 0$. The proof only uses the fact that $\bar{H}_k^i < 1$. Since this is satisfied for the corrected belied, existence follows from the proof of Bose and Daripa (2009), which is an application of Brouwer's fixed point theorem.

**Proposition 4 of Bose and Daripa (2009)** The main result of Bose and Daripa (2009) shows that the seller can extract almost all surplus

**Proposition 6** (Proposition 4 of Bose and Daripa (2009)). *For any preference parameter $\epsilon > 0$, there exists $\delta^*(\epsilon) > 0$ such that for any $\delta < \delta^*(\epsilon)$ and $\eta > 0$, there is an MDM such that in any equilibrium of the game induced by the MDM, the item is sold if at least one buyer has valuation greater than $\eta$ and no type obtains an ex-post surplus greater than $\delta$.*

*Proof.* The first part can be proven as in Bose and Daripa (2009). From the previous results it follows that for any $\epsilon > 0$, there is a $\delta^*(\epsilon) > 0$ such that whenever $\delta < \delta^*(\epsilon)$, an equilibrium exists and all equilibria can be characterized by

$$v_k = p_k + \Delta_k \frac{\bar{H}_k}{1 - \bar{H}_k}.$$

Further, as noted in Section 2.2 for any $\eta \in (0, 1)$, there exists an integer $T$ such that by choosing $n = T$, the price sequence of the MDM covers at least a fraction $(1 - \eta)$ of types. The item is not sold to at most types in $[0, \eta]$. Thus, it only remains to show that no type that buys gets an ex-post surplus greater than $\delta$. This part has to be adjusted.

Since types in $[v_k, v_{k-1}]$ buys at price $p_k$, the ex-post surplus of any type buying at $p_k$ is at most $v_{k-1} - p_k$. Furthermore, the characterization of the equilibrium cut-off types implies

$$
\begin{aligned}
v_{k-1} - p_k &= \Delta_{k-1}\left(1 + \frac{\bar{H}_k}{1 - \bar{H}_k}\right) = \Delta_{k-1}\frac{1}{1 - \bar{H}_k} \\
&= \frac{\epsilon\delta}{2}\underbrace{\left(\frac{1-\delta}{(1-\delta)\frac{\delta\epsilon}{2}}\right)^{k-1}}_{<1}\frac{(1-\epsilon)\overbrace{(F(v_k) + F(v_{k-1}))}^{\leq 2} + \epsilon}{(1-\epsilon)(F(v_{k-1}) - F(v_{k+1})) + \epsilon} \\
&< \frac{\epsilon\delta}{2}\frac{2(1-\epsilon) + \epsilon}{(1-\epsilon)\underbrace{(F(v_{k-1}) - F(v_{k+1}))}_{>0} + \epsilon} \\
&< \frac{\epsilon\delta}{2}\frac{2}{\epsilon} = \delta.
\end{aligned}
$$

Hence, no type can extract an ex-post surplus greater than $\delta$. $\qquad\square$